Three-vector and scalar field identities and uniqueness theorems in Euclidean and Minkowski spaces

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Three-vector and scalar field identities and uniqueness theorems in Euclidean and Minkowski spaces

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Euclidean three-space and Minkowski four-space identities and uniqueness theorems are reviewed and extended. A Helmholtz identity is used to prove two three-vector uniqueness theorems in Euclidean three-space. The first theorem specifies the divergence and curl of the vector, and the second is a Helmholtz type theorem that sums the irrotational and solenoidal parts of the vector. The second theorem is shown to be valid for three-vector fields that are time dependent. A time-dependent extension of the Helmholtz identity is also derived. However, only the three-vector and scalar components of a Minkowski space four-vector identity are shown to yield two identities that lead to a uniqueness theorem of the first or source type. Also, the field equations of this latter theorem appear to be sufficiently general such that the field equations naturally divide into two distinct classes, a four-solenoidal electromagnetic type class in a relativistic transverse gauge and a four-irrotational class in a relativistic longitudinal gauge.
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I. INTRODUCTION

Several recent articles[1–3] have discussed the validity of a Helmholtz theorem on the unique separation of transverse (solenoidal) and longitudinal (irrotational) spatial components for time-dependent three-vector fields. A comment questioning the validity of this Helmholtz theorem[2] appears to be based in part on earlier investigations on the uniqueness of three-vector fields in Minkowski space-time.[4,5] In this paper I will use my work on four-vector uniqueness theorems[6] to confirm the validity of the Helmholtz theorem that Rohrlich[3] applied to time-varying three-vector fields, and to derive new results that verify several results from Refs. 2 and 4. Additional results for time-varying scalar and vector fields will also be obtained.

In Sec. II a review of uniqueness theorems in a Euclidean three-space is presented. One source of the confusion that this article addresses is based on two different but related approaches to three-vector uniqueness theorems. In the first approach a unique three-vector field is obtained by specifying its curl and divergence in terms of a vector and scalar (for example, source) field, respectively. In the second approach, usually referred to as a Helmholtz theorem, a unique three-vector field $\mathbf{F}=\mathbf{F}_{\mathrm{I}}+\mathbf{F}_{\mathrm{S}}$ is represented as a sum of an irrotational part ($\boldsymbol{\nabla}\times\mathbf{F}_{\mathrm{I}}=0$) and a solenoidal part ($\boldsymbol{\nabla}\cdot\mathbf{F}_{\mathrm{S}}=0$). In proving these theorems a vector identity is used that is commonly referred to as a Helmholtz identity. This identity can be derived using a delta function property of a differential operator. For the usual choice of the differential operator as the Laplacian operator it is shown here that only the second of the two uniqueness theorems follows for three-vector fields when an arbitrary time dependence is added.

In Sec. III an extension of the (static) Helmholtz identity to a three-vector field with time dependence is derived. This identity is related to an identity proved in terms of retarded quantities by Heras.[4] The identity derived here expresses the result in terms of a suitable Green's function.

In Sec. IV identities that follow from the three-vector and scalar components of a Minkowski space four-vector Helmholtz identity[6] are obtained. In Sec. V it is shown that this decomposition leads directly to a uniqueness theorem in terms of vector and scalar sources, which implies Maxwell's equations when a suitable covariant gauge is chosen, that is, when the four-divergence of the four-vector potential A^{μ} is set to zero. A second class of field equations is obtained when the four-curl of A^{μ} is set to zero. These results give further weight to the theorem[7] that there are two and only two classes of classical four-vector fields in Minkowski space that are potentially physical when restricted by a relativistically invariant (gauge) constraint.

II. REVIEW OF THREE-VECTOR UNIQUENESS THEOREMS

The history of three-vector uniqueness theorems extends back to Stokes and Helmholtz.[8] More recent treatments of the subject can be found in Sommerfeld,[8] Collin,[9] Plonsey and Collin,[10] Arfken,[11] King,[12] and in Ref. 6.

It is sufficient for the present investigation to take as the volume region of interest the unbounded Euclidean three-space $\mathbb{R}^3$ or, as appropriate, the unbounded Minkowski four-space $\mathbb{R}^{3+1}$. Free space SI units are used throughout.

There are two basic types of three-vector uniqueness theorems over $\mathbb{R}^3$ which are currently known. The first can be stated as follows:[6]

Theorem U1. The divergence and curl of a twice continuously differentiable (static) three-vector field, which vanishes sufficiently rapidly at infinity, uniquely determines the three-vector field over an unbounded volume V of $\mathbb{R}^3$.

In other words, we must specify

$$\boldsymbol{\nabla}\times\mathbf{F}(x,y,z)=\mathbf{j}(x,y,z), \tag{1a}$$

$$\boldsymbol{\nabla}\cdot\mathbf{F}(x,y,z)=\rho(x,y,z), \tag{1b}$$

over the volume V. In an electromagnetic context $\mathbf{j}$ is a circulation current density and ρ is a source charge density. A proof of a version of this theorem over a finite volume of $\mathbb{R}^3$ is given in Ref. 11.

The second uniqueness theorem, which is typically called a Helmholtz theorem, can be stated as follows:[6]

Theorem H1. A general continuous three-vector field defined everywhere in $\mathbb{R}^3$ that along with its first derivatives vanishes sufficiently rapidly at infinity may be uniquely represented as a sum of an irrotational and a solenoidal part, up to a possible additive constant vector.

A proof of theorem H1 is given in Ref. 8. To prove this theorem it is sufficient to prove only that the three-vector $\mathbf{F}$ can be written as[11]

$$\mathbf{F}(x,y,z) = -\nabla\Phi(x,y,z) + \nabla \times \mathbf{A}(x,y,z). \tag{2}$$

Then the three-vector identity

$$\nabla \times \nabla\Phi = 0, \tag{3}$$

and the irrotational field assumption $\nabla \times \mathbf{F}_{\mathrm{I}} = 0$ implies that $\mathbf{F}_{\mathrm{I}} = -\nabla\Phi$ is irrotational. And the three-vector identity

$$\nabla \cdot \nabla \times \mathbf{A} = 0, \tag{4}$$

and the solenoidal field assumption $\nabla \cdot \mathbf{F}_{\mathrm{S}} = 0$ implies that $\mathbf{F}_{\mathrm{S}} = \nabla \times \mathbf{A}$ is solenoidal. Thus Eq. (2) is a sum of an irrotational and a solenoidal part.

The most straightforward proof of Eq. (2) (see Refs. 9 and 10) and therefore of theorem H1 is to obtain a Helmholtz identity[8] that is of the same form as Eq. (2). A proof of this identity is presented here so that our analysis is self contained. This proof is based on the assumption that there exists a solution for a three-vector $\mathbf{F}$ of an inhomogeneous vector Poisson equation in Cartesian coordinates, and that each separate Cartesian component of $\mathbf{F}$ is a solution of a scalar Poisson equation. The scalar Poisson equation can then be solved in terms of a two-point scalar Green's function $G(\mathbf{r},\mathbf{r}')$ that connects its unit delta function source located at the source point $\mathbf{r}' = (x',y',z')$ to a measurement at the field point $\mathbf{r} = (x,y,z)$. Because

$$\nabla^2 \frac{1}{4\pi r} = -\delta^3(\mathbf{r} - \mathbf{r}'), \tag{5}$$

for all $\mathbf{r}$ and $\mathbf{r}'$ with $r \equiv |\mathbf{r} - \mathbf{r}'|$, the Poisson equation defining the Green's function,

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = -\delta^3(\mathbf{r} - \mathbf{r}'), \tag{6}$$

leads to $G(\mathbf{r},\mathbf{r}') = 1/4\pi r$. The delta function property of the vector identity (5) can be applied to any well-behaved function $\mathbf{F}(x,y,z)$ as

$$\mathbf{F}(x,y,z) = \int_{V'} \mathbf{F}(x',y',z')\delta^3(\mathbf{r} - \mathbf{r}')dV' \tag{7a}$$

$$= \int_{V'} \mathbf{F}(x',y',z')\nabla^2\left(\frac{-1}{4\pi r}\right)dV'. \tag{7b}$$

Because the Laplacian operator acts only on the field point coordinates, it can be brought outside of the integration over the source point coordinates. We can use the well known three-space identity

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) \tag{8}$$

to rewrite Eq. (7b) as

$$\mathbf{F}(x,y,z) = -\nabla \int_{V'} \nabla \cdot \frac{\mathbf{F}(x',y',z')}{4\pi r} dV' + \nabla \times \int_{V'} \nabla \frac{\mathbf{F}(x',y',z')}{4\pi r} dV', \tag{9}$$

where one of the ∇ operators has been brought back into each of the integrals, which is allowed because they operate only on the field coordinates. Although Eq. (9) is in a form like Eq. (2) as required to prove theorem H1, it is necessary to obtain an identity that is suitable to prove the three-space theorem U1 as well. If we use the vector identities[10]

$$\nabla \cdot (\phi\mathbf{A}) = \mathbf{A} \cdot \nabla\phi + \phi\nabla \cdot \mathbf{A}, \tag{10a}$$

$$\nabla \times (\phi\mathbf{A}) = -\mathbf{A} \times \nabla\phi + \phi\nabla \times \mathbf{A}, \tag{10b}$$

(note that the divergence and curl equal zero with respect to the field coordinates of a vector $\mathbf{A}$ which is a function only of the source coordinates) and use the identity $\nabla(1/r) = -\nabla'(1/r)$, we find

$$\mathbf{F}(x,y,z) = \nabla \int_{V'} \mathbf{F}(x',y',z') \cdot \nabla' \frac{1}{4\pi r} dV' + \nabla \times \int_{V'} \mathbf{F}(x',y',z') \times \nabla' \frac{1}{4\pi r} dV'. \tag{11}$$

The vector identities (10) are now applied again, this time on the integrands of Eq. (11) where the ∇ operators act on the source coordinates. We obtain four terms; the ones containing $\nabla \cdot (\phi\mathbf{A})$ and $\nabla \times (\phi\mathbf{A})$, respectively, become surface integrals via the following identities,[10] (that is, letting $\mathbf{T} = \phi\mathbf{A}$)

$$\int_V \nabla \cdot \mathbf{T}\, dV = \oint_S \mathbf{T} \cdot d\mathbf{S} = \oint_S \mathbf{T} \cdot \mathbf{n}\, dS, \tag{12a}$$

$$\int_V \nabla \times \mathbf{T}\, dV = \oint_S \mathbf{n} \times \mathbf{T}\, dS = -\oint_S \mathbf{T} \times \mathbf{n}\, dS, \tag{12b}$$

where $\mathbf{n}$ is the unit surface normal of S bounding V. These integrals vanish as $r \to \infty$ for the field $\mathbf{F}$, which is assumed to fall off sufficiently rapidly at infinity. The final two terms are now of the proper form yielding the desired identity

$$\mathbf{F}(x,y,z) = -\nabla \int_{V'} \frac{\nabla' \cdot \mathbf{F}(x',y',z')}{4\pi r} dV' + \nabla \times \int_{V'} \frac{\nabla' \times \mathbf{F}(x',y',z')}{4\pi r} dV' \tag{13}$$

over all Euclidean three-space.

A proof of Eq. (13) can be traced back to Stokes.[8] Nevertheless, Eq. (13) is sometimes referred to as a "Helmholtz identity" (see Ref. 10). For an extended version of Eq. (13) in a finite volume of $\mathbb{R}^3$ the surface integral terms can be retained.[6,9,10] Equation (13) is of the form of Eq. (2) and can therefore be considered as completing the proof of theorem H1, provided that the integrals are well defined. For the integrals to be well defined we must assume that the field $\mathbf{F}$ vanishes sufficiently rapidly at infinity. As noted in Ref. 6, passing the remaining vector derivatives over the field point coordinates for a twice continuously differentiable vector field $\mathbf{F}$ into each of the respective integrals of Eq. (13) can improve the convergence properties of the integrands.

To prove theorem U1 using the Helmholtz identity (13) for the (static) three-vector field $\mathbf{F}(\mathbf{r})$ we postulate the existence of a second three-vector field $\mathbf{G}(\mathbf{r})$, which also satisfies Eqs. (1) and (13). That is, we replace $\mathbf{F}(\mathbf{r})$ on the left-hand side of Eqs. (1a) and (1b) by $\mathbf{G}(\mathbf{r})$, leaving the right-hand side of Eqs. (1a) and (1b) unchanged. The three-vector field $\mathbf{F}(\mathbf{r})$ is unique if we can show that

$$\mathbf{W}(\mathbf{r}) \equiv \mathbf{F}(\mathbf{r}) - \mathbf{G}(\mathbf{r}) = 0. \tag{14}$$

If we take the divergence of $\mathbf{W}$ and use Eq. (1b), we obtain

$$\boldsymbol{\nabla} \cdot \mathbf{W} = \boldsymbol{\nabla} \cdot \mathbf{F} - \boldsymbol{\nabla} \cdot \mathbf{G} = \rho - \rho = 0 \tag{15}$$

for all $\mathbf{r}$ in $\mathbb{R}^3$. We next take the curl of $\mathbf{W}$ and use Eq. (1a) to obtain

$$\boldsymbol{\nabla} \times \mathbf{W} = \boldsymbol{\nabla} \times \mathbf{F} - \boldsymbol{\nabla} \times \mathbf{G} = \mathbf{j} - \mathbf{j} = 0 \tag{16}$$

for all $\mathbf{r}$ in $\mathbb{R}^3$. The substitution of the results (15) and (16) for the three-vector field $\mathbf{W}$ into the Helmholtz identity (13) yields the result $\mathbf{W}(\mathbf{r})=0$, which implies [via the definition of $\mathbf{W}(\mathbf{r})$ in Eq. (14)] that $\mathbf{F}(\mathbf{r})=\mathbf{G}(\mathbf{r})$ everywhere in $\mathbb{R}^3$. This proof implies that the (static) three-vector field $\mathbf{F}$ is uniquely determined by Eqs. (1a) and (1b), thus proving theorem U1 on the uniqueness of (static) three-vector fields in terms of their curl and divergence, that is, in terms of a vector and a scalar (source) field, respectively. [For a finite volume of $\mathbb{R}^3$ the normal components $\mathbf{W}\cdot\mathbf{n}$ and tangential components $\mathbf{W} \times \mathbf{n}$ also vanish in a similar fashion via a surface charge density σ and surface current density $\mathbf{K}$, as can be shown for example in an explicit calculation for the case of a massive (static) three-vector field.[6]]

We now state the obvious. Equations (1a) and (1b) associated with theorem U1 are not equivalent to Maxwell's equations, which uniquely specify the electromagnetic fields. They do not contain any partial time derivative terms and do not contain any coupling between different three-vector fields. Therefore, Eqs. (1a) and (1b) and theorem U1 do not hold for time-varying electric and magnetic fields, which are examples of *hyperbolically propagating*[4] time-varying three-vector fields. Therefore, theorem U1 does not hold for all time-varying three-vector fields because there is at least one counterexample, that is, electromagnetism.

In contrast, the Helmholtz theorem H1 is quite different in scope. It is essentially a projection theorem which uses three-vector analysis to project out the longitudinal (irrotational) parts (which have zero curl) and transverse (solenoidal) parts (which have zero divergence) of an arbitrary three-vector field, and states that any three-vector field can be represented as a sum of these two parts. To show that this theorem holds even for time-dependent three-vectors, it is sufficient to verify the steps employed in deriving the Helmholtz identity (13), because it is of the proper form of Eq. (2), which is sufficient to prove theorem H1. Following this line of reasoning, first observe that the delta function integral property in Eq. (7b) holds even for well-behaved time-varying three-vector functions $\mathbf{F}(x,y,z,t)$ because the space and time variables are independent, and consequently space and time integrations are performed separately. As for Eq. (8), it is a three-vector identity over the Euclidean vector space $\mathbb{R}^3$ and over the associated scalars in the real number field $\mathbb{R}$ which applies to any (even time-dependent) three-vector because it is based on multiplication of vectors by scalars (a defining property of a vector space) and on the two principal ways that vectors can be multiplied by each other in $\mathbb{R}^3$, that is, dot and cross products. [More formally, for $\mathbf{A}$ as a time-dependent variable, Eq. (8) forms a one-parameter family of vector fields parametrized by t. For each time t, Eq. (8) holds. The same can be said for Eq. (7b).] Equation (8) is often used in deriving the spatial parts of wave equations and is the principal vector identity that leads to the decomposition (2). For the same reasons the vector identities (10) and integral identities (12) are valid for time-varying fields. Therefore, the Helmholtz identity (13) holds even for time-varying three-vector fields and so Helmholtz's theorem H1 follows for them as well via Eq. (2).

Helmholtz's theorem H1 has been used frequently with time-varying fields in both classical and quantum mechanical contexts. Rohrlich[3] has demonstrated that this Helmholtz decomposition holds even for electromagnetic sources, provided that the sources are bounded in space. This latter requirement is essentially equivalent to the case of the general fields discussed here which are assumed to vanish sufficiently rapidly at infinity in unbounded $\mathbb{R}^3$. The relevant requirement in a bounded volume V of $\mathbb{R}^3$ is that the vector (or scalar) field must be sufficiently smooth,[6] that is, a twice continuously differentiable function on the union of V and its bounding surface S, in order that a Helmholtz identity in a finite volume and its associated finite volume Helmholtz theorem is satisfied.[6,9,10] Equation (13) has also been used in a time-varying context to demonstrate its compatibility with electromagnetism in the Coulomb gauge and to derive the Aharonov-Bohm transverse vector potential.[13–16] The Helmholtz theorem was recently used to prove a vector identity for the volume integral of the square of a vector field which could be time varying, and Eq. (13) was used to derive two expressions for the energy of the electromagnetic field.[17]

III. A TIME-DEPENDENT HELMHOLTZ IDENTITY

A logical next step for a three-vector field that is required to propagate via an inhomogeneous hyperbolic wave equation in terms of its three-vector (current) source is to derive an extension of the three-vector Helmholtz identity by choosing the d'Alembertian wave equation operator as the differential operator used to obtain an associated delta function identity. If we assume Cartesian coordinates, it is sufficient to solve an inhomogeneous scalar d'Alembertian wave equation in terms of a two-point scalar Green's function $G(x^\nu,x'^\nu)$ which connects its unit delta function source located at the space-time source point x'^ν to a measurement at the space-time field point $x^\nu=(ct,x,y,z)$ in $\mathbb{R}^{3+1}$. That is,

$$\Box G(x^\nu,x'^\nu) = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(x^\nu,x'^\nu) \tag{17a}$$

$$= -\,\delta(x^0 - x'^0)\,\delta^3(\mathbf{r} - \mathbf{r}')$$
$$= -\,\delta^{(4)}(x^\nu - x'^\nu). \tag{17b}$$

An example of a Green's function that satisfies Eq. (17b) assuming timelike causality is the familiar retarded Green's function as derived for example by Cushing[18] or Jackson,[19] which for the metric signature $(-+++)$ used in this article is

$$G_{\mathrm{ret}}(\mathbf{r},\mathbf{r}';t,t') = \frac{1}{4\pi}\frac{\delta(|\mathbf{r}-\mathbf{r}'| - c(t-t'))}{|\mathbf{r}-\mathbf{r}'|}, \tag{18}$$

for $t > t'$. The retarded Green's function (18) is derived via a spectral decomposition of the delta function, taking into account homogeneous boundary conditions on a closed spatial surface, as well as timelike causality with the initial conditions $G(\mathbf{r},\mathbf{r}';t,t')=0$ and $\partial G(\mathbf{r},\mathbf{r}';t,t')/\partial t=0$ for $t<t'$, and the Green's function symmetry relation

$$G(\mathbf{r},\mathbf{r}';t,t') = G(\mathbf{r}',\mathbf{r};-t',-t). \tag{19}$$

For a relativistically invariant version of Eq. (18), that is, $G_{\mathrm{ret}}(x,x')$, and for the advanced Green's function $G_{\mathrm{adv}}(x,x')$, see, for example, Ref. 6. (In what follows the usual shorthand notation is adopted, and the superscripts in the functional dependencies are dropped for brevity.) The spectral decomposition of the four-space delta function of Eq. (17b) also yields certain required covariant and contravariant derivative properties of $G(x,x')$ in Minkowski space:[6]

$$\partial_\mu G(x,x') = -\partial'_\mu G(x,x'), \tag{20a}$$

$$\partial^\mu G(x,x') = -\partial'^\mu G(x,x'). \tag{20b}$$

With $\partial_\mu=((1/c)\partial/\partial t,\boldsymbol{\nabla})$ and $\partial^\mu=(-(1/c)\partial/\partial t,\boldsymbol{\nabla})$ we obtain the space and time components of Eq. (20a) as

$$\partial_i G(x,x') = -\partial'_i G(x,x'), \tag{21a}$$

$$\partial_0 G(x,x') = -\partial'_0 G(x,x'), \tag{21b}$$

with equivalent relations for the contravariant derivatives. Note that Eq. (21a) is analogous to the relation $\boldsymbol{\nabla}(1/r) = -\boldsymbol{\nabla}'(1/r)$ used in $\mathbb{R}^3$, which can be rewritten as $\boldsymbol{\nabla}G(\mathbf{r},\mathbf{r}')=-\boldsymbol{\nabla}'G(\mathbf{r},\mathbf{r}')$. In $\mathbb{R}^{3+1}$ there is the additional time derivative property (21b). For the present analysis Eqs. (21a) and (21b) can be written simply as

$$\boldsymbol{\nabla}G(x,x') = -\boldsymbol{\nabla}'G(x,x'), \tag{22a}$$

$$\frac{\partial G(x,x')}{\partial t} = -\frac{\partial G(x,x')}{\partial t'}. \tag{22b}$$

The delta function property (17b) can be applied for any well-behaved function $\mathbf{F}(x^\nu)$ as an integration over the unbounded four-volume V'_4 of $\mathbb{R}^{3+1}$ as follows:

$$\mathbf{F}(x) = \int_{V'_4} \mathbf{F}(x')\delta^{(4)}(x-x')d^4x' \tag{23a}$$

$$= \int_{V'_4} \mathbf{F}(x')\Box(-G(x,x'))dV'cdt'. \tag{23b}$$

It is not necessary at this point to specify the choice of a particular Green's function $G(x,x')$, other than that it satisfies Eq. (17b). In fact, there is a distinct advantage to not choosing a retarded (advanced) Green's function for doing the time integration at this point because the resulting retarded (advanced) three-vector fields are much more difficult with which to work. In contradistinction, the space and time derivatives of a three-vector with unmixed arguments, for example, $\mathbf{F}(x')=\mathbf{F}(ct',x',y',z')$, need no special treatment. Consequently, the entire (spatial) derivation of the Helmholtz identity (13), in view of the equivalent form of relation (22a), can be used with Eq. (23b), yielding

$$\begin{aligned}\mathbf{F}(x) = &-\boldsymbol{\nabla}\int_{V'_4}\boldsymbol{\nabla}'\cdot\mathbf{F}(x')G(x,x')dV'cdt' \\ &+\boldsymbol{\nabla}\times\int_{V'_4}\boldsymbol{\nabla}'\times\mathbf{F}(x')G(x,x')dV'cdt' \\ &+\frac{1}{c^2}\frac{\partial}{\partial t}\int_{V'_4}\frac{\partial(\mathbf{F}(x')G(x,x'))}{\partial t}dV'cdt',\end{aligned} \tag{24}$$

where one of the unprimed partial time derivatives has been taken outside of the integration over the primed time derivative, and the other has been put in front of $\mathbf{F}(x')$. The application of the time derivative in the integrand of the third term of Eq. (24) gives

$$\frac{\partial(\mathbf{F}(x')G(x,x'))}{\partial t} = -\mathbf{F}(x')\frac{\partial G(x,x')}{\partial t'} \tag{25a}$$

$$= -\frac{\partial(\mathbf{F}(x')G(x,x'))}{\partial t'} + \frac{\partial\mathbf{F}(x')}{\partial t'}G(x,x'), \tag{25b}$$

where the unprimed time derivative of $\mathbf{F}(x')$ is zero and Eq. (22b) has been used in Eq. (25a). The time integration of the first term of Eq. (25b) can be assumed to vanish as $t' \to \pm\infty$ for the field $\mathbf{F}$ which is assumed to be bounded in time. Therefore, only the second term of Eq. (25b) contributes to Eq. (24), reducing it to

$$\begin{aligned}\mathbf{F}(x) = &-\boldsymbol{\nabla}\int_{V'_4}\boldsymbol{\nabla}'\cdot\mathbf{F}(x')G(x,x')dV'cdt' \\ &+\boldsymbol{\nabla}\times\int_{V'_4}\boldsymbol{\nabla}'\times\mathbf{F}(x')G(x,x')dV'cdt' \\ &+\frac{1}{c^2}\frac{\partial}{\partial t}\int_{V'_4}\frac{\partial\mathbf{F}(x')}{\partial t'}G(x,x')dV'cdt',\end{aligned} \tag{26}$$

which can be regarded as a time-dependent generalization of the (static) Helmholtz identity (13) for any suitable Green's function $G(x,x')$ that is a solution of Eq. (17b). [A result like Eq. (26), but in terms of a retarded Green's function, was given in Ref. 2 with a comment that the proof followed from Eq. (23b); we have verified this proof in detail here.] If we substitute the retarded Green's function (18) into Eq. (26), we obtain

$$\begin{aligned}\mathbf{F}(x) = &-\boldsymbol{\nabla}\int_{V'_4}\boldsymbol{\nabla}'\cdot\mathbf{F}(x')\frac{\delta(r-c(t-t'))}{4\pi r}dV'cdt' \\ &+\boldsymbol{\nabla}\times\int_{V'_4}\boldsymbol{\nabla}'\times\mathbf{F}(x')\frac{\delta(r-c(t-t'))}{4\pi r}dV'cdt' \\ &+\frac{1}{c^2}\frac{\partial}{\partial t}\int_{V'_4}\frac{\partial\mathbf{F}(x')}{\partial t'}\frac{\delta(r-c(t-t'))}{4\pi r}dV'cdt'.\end{aligned} \tag{27}$$

When the time integration in Eq. (27) is performed, the argument of the delta function implies that all t' dependent variables in the integrand are evaluated at $t-r/c$. If we use a retarded potential bracket notation where $[\mathbf{P}(\mathbf{r}',ct')]_{\rm ret}$ denotes evaluation at the retarded time $t'=t-r/c$, then Eq. (27) can be rewritten as

$$\mathbf{F}(x) = -c\,\boldsymbol{\nabla}\int_{V'}\frac{[\boldsymbol{\nabla}'\cdot\mathbf{F}(\mathbf{r}',ct')]_{\rm ret}}{4\pi r}dV' + c\,\boldsymbol{\nabla}\times\int_{V'}\frac{[\boldsymbol{\nabla}'\times\mathbf{F}(\mathbf{r}',ct')]_{\rm ret}}{4\pi r}dV' + \frac{1}{c}\frac{\partial}{\partial t}\int_{V'}\left[\frac{\partial\mathbf{F}(\mathbf{r}',ct')}{\partial t'}\right]_{\rm ret}\frac{1}{4\pi r}dV', \tag{28}$$

which is similar to Eq. (13) in Ref. 4. The result (28) is obtained here in a more easily verified manner because tricky retarded potential calculations are avoided by retaining the general Green's function $G(x,x')$ through to the time-dependent Helmholtz identity (26).

With suitable substitutions the identity (26) can be rewritten in a form resembling Eq. (2) as

$$\mathbf{E}(ct,x,y,z) = -\boldsymbol{\nabla}\phi - \boldsymbol{\nabla}\times\mathbf{K} - \frac{\partial\mathbf{A}}{\partial t}, \tag{29}$$

where a minus sign has been introduced in the substitutions for the second and third terms of Eq. (29) to emphasize its similarity to a well-known definition of the electric field in terms of the electric and magnetic potentials.[20] A compatible definition of the magnetic field[20] with appropriate variable and sign changes would be

$$\mathbf{B}(ct,x,y,z) = -\boldsymbol{\nabla}\xi + \boldsymbol{\nabla}\times\mathbf{A} - \frac{\partial\mathbf{K}}{\partial t}. \tag{30}$$

In a different calculation on a related subject, Eq. (13) in Ref. 4 [similar to Eq. (28) here] was used to derive an extension of Jefimenko's time-dependent generalizations of the Coulomb and Biot-Savart laws[21] to include magnetic charge and current sources. The time-dependent Helmholtz identity (26) therefore has interesting applications.

Although Eq. (29) is an example of the generalization of Eq. (2) via the time-dependent Helmholtz identity (26), it is the possibility of using Eq. (26) to obtain uniqueness theorems that is of interest here. Consider the claim by Heras[2] that the terms of Eq. (26) could be labeled (preserving the order of the terms) as follows:

$$\mathbf{E}(ct,x,y,z) = \mathbf{E}_{\parallel} + \mathbf{E}_{\perp} + \mathbf{E}_{\rm T}. \tag{31}$$

The longitudinal component $\mathbf{E}_{\parallel}$ is irrotational and satisfies

$$\nabla\times\mathbf{E}_{\parallel} = 0, \tag{32}$$

and the transverse component $\mathbf{E}_{\perp}$ is solenoidal and satisfies

$$\boldsymbol{\nabla}\cdot\mathbf{E}_{\perp} = 0. \tag{33}$$

The time component $\mathbf{E}_{\rm T}$ may have both transverse and longitudinal components. The first term of Eq. (26) is a longitudinal component via Eq. (3) and the second term of Eq. (26) is a transverse component via Eq. (4). However, Heras[4] states in this context that "the specification of $\partial\mathbf{F}/\partial t$ is not available in general and there is not a simple approach for obtaining it."

The salient issue here is that the divergence and curl operations are sufficient to produce all the possible projections of vector fields in $\mathbb{R}^3$. That is, the application of a time partial derivative cannot project out a time component longitudinal to the time dimension from a three-vector field that has only spatial dimensions. Four-vector fields are required for that. All that has been done here is that an independent (time) variable has been included, without forming a vector space of one more dimension. The absence of a unique specification of a time projection for the hypothetical time component $\mathbf{E}_{\rm T}$ belonging to the three-vector space $\mathbb{R}^3$ makes it problematic for there to be a time-dependent version of theorem H1 based on Eq. (31) [or Eq. (26)].

Neither does Eq. (29), as a representative example of a choice of variables following from Eq. (26), appear to be adequate to generate a new uniqueness theorem of the type of theorem U1. To demonstrate this deficiency we set the divergence of Eq. (29) equal to a suitable (electric) scalar source charge density $\rho_{\rm e}$ scaled by the free space permittivity constant ϵ_0 yielding

$$\frac{\rho_{\rm e}}{\epsilon_0} = \boldsymbol{\nabla}\cdot\mathbf{E} = -\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\phi - \frac{\partial\boldsymbol{\nabla}\cdot\mathbf{A}}{\partial t}, \tag{34}$$

where the second term of Eq. (29) vanishes via Eq. (4). Then we set the curl of Eq. (29) equal to a suitable (magnetic) three-vector current density $\mathbf{j}_{\rm m}$ scaled by the free space permeability constant μ_0 yielding

$$-\mu_0\mathbf{j}_{\rm m} = \boldsymbol{\nabla}\times\mathbf{E} = -\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{K} - \frac{\partial\boldsymbol{\nabla}\times\mathbf{A}}{\partial t}, \tag{35}$$

where the first term of Eq. (29) vanishes via Eq. (3). Then we take the partial time derivative of Eq. (29) yielding

$$\frac{\partial\mathbf{E}}{\partial t} = -\boldsymbol{\nabla}\frac{\partial\phi}{\partial t} - \frac{\partial\boldsymbol{\nabla}\times\mathbf{K}}{\partial t} - \frac{\partial^2\mathbf{A}}{\partial t^2}. \tag{36}$$

In contrast to Eqs. (34) and (35) it is not evident what source density should be equated to the left-hand side of Eq. (36). In the present context of a time-varying electromagnetism with both electric and magnetic charges, such a third source relation would be inconsistent because the remaining two sources, $\rho_{\rm m}$ and $\mathbf{j}_{\rm e}$, would need to be accounted for by the compatible relation (30). In addition, although Eq. (34) is adequate for the time-varying case, Eq. (35) is adequate only for static fields [compare with Eq. (11) in Ref. 4]. If there are no magnetic sources, $\mathbf{j}_{\rm m}$, $\rho_{\rm m}$, ξ, and $\mathbf{K}$ vanish, and then Eq. (35) also lacks a source to uniquely specify it (or is reduced to no more than the static case by setting $\mathbf{j}_{\rm m}=0$). Equation (30) suffers from similar difficulties with relations analogous to Eqs. (35) and (36). Consequently, it is problematic for there to be a time-dependent version of theorem U1 based on Eqs. (29) and (30) [or Eq. (26)] for the most general electromagnetic case.

IV. TIME-DEPENDENT THREE-VECTOR AND SCALAR FIELD IDENTITIES

In Sec. III it was shown that the use of the time-dependent generalization (26) of the (static) Helmholtz identity (13) does not appear to be adequate to obtain time-dependent versions of uniqueness theorems U1 or H1. However, an "extended curl"[5] approach for obtaining a uniqueness theorem for a pair of coupled time-dependent three-vector fields in terms of two scalar and two three-vector source fields, and in terms of two auxiliary field definitions such as Eqs. (29) and (30), was formulated in Ref. 22. This latter theorem was referred to there as a generalized Helmholtz theorem.[22] From our point of view the first part of this two part theorem is of type U1, and the second part is of the Helmholtz (projection) type H1, although the concepts of irrotational and solenoidal are omitted. The proof is based on the existence of nontrivial solutions of the associated elliptical or hyperbolic differential equations. In the hyperbolic case the field equations can be used to treat the electromagnetic case with both electric and magnetic charges. As interesting as this theorem is, its equations are not derived from first principles, but are postulated in their entirety. Secondly, it is a theorem for three-vectors in $\mathbb{R}^3$ with the time dependence put in by hand, that is, it is not manifestly covariant. It will be shown here that a manifestly covariant approach[6] in $\mathbb{R}^{3+1}$ can yield the desired time-varying uniqueness theorems that do not appear to follow from first principles in $\mathbb{R}^3$.

It has been previously shown (by taking a static Newtonian limit) that the spatial components of the identity Eq. (69) of theorem II of Ref. 6 [see Eq. (37)] is a Minkowski space generalization of the (Euclidean three-space) Helmholtz identity (13), but more generally in a finite volume of $\mathbb{R}^3$. In addition, the fourth or scalar field component of this identity [see (Eq. (45))] was shown to yield a scalar field identity, Eq. (96) of Ref. 6, in a similar static Newtonian limit. To obtain the full time-dependent results we first take the space components of this identity (that is, where the four-space index μ is replaced by the three-space index j):

$$A^j(x) = -\partial^j\left[\int_{V_4'} \partial_\nu' A^\nu(x')G(x,x')d^4x' - \oint_{\Sigma'} (A^\nu(x')n_\nu')G(x,x')d\Sigma'\right] - \partial_\alpha\left[\int_{V_4'} (\partial'^\alpha A^j(x') - \partial'^j A^\alpha(x'))G(x,x')d^4x' + \oint_{\Sigma'} (A^\alpha(x')n'^j - A^j(x')n'^\alpha)G(x,x')d\Sigma'\right], \quad (37)$$

where the four-volume region V_4' of Minkowski space-time is bounded by the three-surface Σ', and the unprimed derivatives have been factored out of the primed coordinate integrals. If we adopt three-vector notation and allow the four-volume region V_4' to expand to include all Minkowski space-time so that the three-surface integrals can be dropped [under the assumption that the four-vector field $A^\mu(x)=(\phi/c,\mathbf{A})$ vanishes sufficiently rapidly at infinity], we can simplify Eq. (37) to

$$\mathbf{A}(x) = -\boldsymbol{\nabla}\int_{V_4'}\left(\boldsymbol{\nabla}'\cdot\mathbf{A}(x') + \frac{1}{c^2}\frac{\partial\phi(x')}{\partial t'}\right)G(x,x')d^4x' - \frac{1}{c}\frac{\partial}{\partial t}\int_{V_4'}\left(-\frac{1}{c}\frac{\partial\mathbf{A}(x')}{\partial t'} - \frac{\boldsymbol{\nabla}'\phi(x')}{c}\right)G(x,x')d^4x' + \boldsymbol{\nabla}\times\int_{V_4'}(\boldsymbol{\nabla}'\times\mathbf{A}(x'))G(x,x')d^4x', \quad (38)$$

where the first four-volume integral of Eq. (38) follows from the first four-volume integral of Eq. (37), the second four-volume integral of Eq. (38) follows from the $\alpha=0$ component of the second four-volume integral of Eq. (37), and the third four-volume integral of Eq. (38) follows from that same term's $\alpha=i$ spatial components by comparing the implied sum on i for each of the $j=1,2,3$ components with the components of the vector triple cross product in Eq. (38). If we make the change of variables [using the components of the four-curl of A^μ, that is, Maxwell's field tensor $F^{\mu\nu}$, see Eqs. (55) and (56)] with

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla\times\mathbf{A}, \quad (39)$$

and the change of variables [using the four-divergence of A^μ, see Eq. (57)] with

$$C = \boldsymbol{\nabla}\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}, \quad (40)$$

we can reduce Eq. (38) to

$$\mathbf{A}(x) = -\boldsymbol{\nabla}\int_{V_4'} C(x')G(x,x')d^4x' - \frac{1}{c^2}\frac{\partial}{\partial t}\int_{V_4'}\mathbf{E}(x')G(x,x')d^4x' + \boldsymbol{\nabla}\times\int_{V_4'}\mathbf{B}(x')G(x,x')d^4x'. \quad (41)$$

We can move the unprimed $\boldsymbol{\nabla}$ operator into the first (primed) integral of Eq. (41), and then as for Eq. (25), we can express the argument of the integral as

$$\boldsymbol{\nabla}(C(x')G(x,x')) = -C(x')\boldsymbol{\nabla}'G(x,x') \quad (42a)$$

$$= -\boldsymbol{\nabla}'(C(x')G(x,x')) + (\boldsymbol{\nabla}'C(x'))G(x,x'), \quad (42b)$$

because the unprimed gradient of $C(x')$ is zero and the Green's function property (22a) is used in Eq. (42a). Note that

$$\int_{V'}\boldsymbol{\nabla}'(C(x')G(x,x'))dV' = 0 \quad (43)$$

over the infinite three-volume V' for the scalar field $C(x')$, which is assumed to vanish sufficiently rapidly at infinity because $\mathbf{A}$ and ϕ are assumed to do so also. Therefore, only the second term of Eq. (42b) contributes to Eq. (41). If we use the result (25b) on the second integral of Eq. (41), assume that $\mathbf{E}$ is bounded in time, and use Eq. (10b) and the surface integral argument that led to Eq. (13) on the third integral of Eq. (41), we obtain the desired three-vector field identity

$$\mathbf{A}(x) = \int_{V_4'}\left(-\boldsymbol{\nabla}'C(x') - \frac{1}{c^2}\frac{\partial\mathbf{E}(x')}{\partial t'} + \boldsymbol{\nabla}'\times\mathbf{B}(x')\right)G(x,x')d^4x'. \quad (44)$$

In a similar fashion a scalar component of the identity Eq. 69 of Ref. 6 can be obtained as follows:

$$A^0(x) = -\partial^0\left[\int_{V_4'} \partial_\nu' A^\nu(x')G(x,x')d^4x' - \oint_{\Sigma'} (A^\nu(x')n_\nu')G(x,x')d\Sigma'\right] - \partial_\alpha\left[\int_{V_4'} (\partial'^\alpha A^0(x') - \partial'^0 A^\alpha(x'))G(x,x')d^4x' + \oint_{\Sigma'} (A^\alpha(x')n'^0 - A^0(x')n'^\alpha)G(x,x')d\Sigma'\right]. \tag{45}$$

We adopt three-vector notation, allow the four-volume V_4' to expand to include all Minkowski space-time, and set $A^0 = (\phi/c)$ so that Eq. (45) simplifies to

$$\frac{\phi(x)}{c} = \frac{1}{c}\frac{\partial}{\partial t}\int_{V_4'} \left(\nabla' \cdot \mathbf{A}(x') + \frac{1}{c^2}\frac{\partial\phi(x')}{\partial t'}\right)G(x,x')d^4x' - \nabla \cdot \int_{V_4'} \left(\frac{\nabla'\phi(x')}{c} + \frac{1}{c}\frac{\partial \mathbf{A}(x')}{\partial t'}\right)G(x,x')d^4x'. \tag{46}$$

The same change of variables, Eqs. (39) and (40), reduces Eq. (46) to

$$\frac{\phi(x)}{c} = \frac{1}{c}\frac{\partial}{\partial t}\int_{V_4'} C(x')G(x,x')d^4x' + \frac{1}{c}\nabla \cdot \int_{V_4'} \mathbf{E}(x')G(x,x')d^4x'. \tag{47}$$

We can move the unprimed $\partial/\partial t$ operator into the first (primed) integral of Eq. (47), and then, similar to Eq. (25), we can express the integrand as

$$\frac{\partial(C(x')G(x,x'))}{\partial t} = -C(x')\frac{\partial G(x,x')}{\partial t'} \tag{48a}$$

$$= -\frac{\partial(C(x')G(x,x'))}{\partial t'} + \frac{\partial C(x')}{\partial t'}G(x,x'), \tag{48b}$$

because the unprimed time derivative of $C(x')$ is zero; and the Green's function property (22b) was used in Eq. (48a). A time integration of the first term of Eq. (48b) can be assumed to vanish as $t' \to \pm\infty$ for the field C, which is assumed to be bounded in time because $\mathbf{A}$ and ϕ are assumed to do so also. Therefore, only the second term of Eq. (48b) contributes to Eq. (47). We use Eq. (10a) and the surface integral argument which led to Eq. (13) on the second integral of Eq. (47) to obtain the desired scalar field identity

$$\frac{\phi(x)}{c} = \frac{1}{c}\int_{V_4'} \left(\frac{\partial C(x')}{\partial t'} + \nabla' \cdot \mathbf{E}(x')\right)G(x,x')d^4x'. \tag{49}$$

V. UNIQUENESS THEOREMS FOR THREE-VECTOR AND SCALAR FIELDS IN MINKOWSKI SPACE

The Minkowski space Helmholtz identity Eq. 69 of Ref. 6, which combines Eqs. (37) and (45), can be written as[6]

$$A^\mu = \partial_\alpha A^{\alpha\mu} + \partial^\mu A, \tag{50}$$

with suitable definitions for $A^{\alpha\mu}$ and A. A four-curl formed from the second term of Eq. (50) is zero by

$$\partial^\mu(\partial^\nu A) - \partial^\nu(\partial^\mu A) = 0, \tag{51}$$

and so the second term of Eq. (50) is four-irrotational.[6] Also, the four-divergence of the first term of Eq. (50) is zero by

$$\partial_\mu(\partial_\alpha A^{\alpha\mu}) = 0, \tag{52}$$

because it is a contraction of a symmetric factor $\partial_\mu\partial_\alpha$ and an antisymmetric factor $A^{\alpha\mu}$; hence the first term of Eq. (50) is four-solenoidal.[6] In Ref. 6 I used Eqs. (50), (51), and (52) to prove theorem X of Ref. 6, which generalizes the Euclidean three-space theorem H1 to any finite or entire volume of Minkowski space. By arguments paralleling those in Sec. II of this article,[6] I also proved theorem V of Ref. 6, a Minkowski space generalization of theorem U1 (for any finite or entire volume of $\mathbb{R}^{3+1}$).

All that remains is to state a meaningful uniqueness theorem of the source type, with a scalar and a vector source, which under a suitable covariant constraint covers the theory of electromagnetism. The resulting theorem is not entirely analogous to theorem U1 because it involves coupled fields. But it may be the best that we can do, being in a sense based on a projection of a most general Minkowski four-space Helmholtz identity for $A^\mu(x)$ into Euclidean three-space components. The theorem associates the arguments of the integrals of the three-vector components (44) and scalar component (49) as derived from the four-vector field Helmholtz identity of Sec. IV with the arguments of the inhomogeneous wave equation Green's function integrals [see Eq. (60)] as follows.

Theorem U2. Given the twice continuously differentiable time-varying three-vector fields $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{A}$, and scalar fields C and ϕ as defined by

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{53a}$$

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{53b}$$

$$C = \nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}, \tag{53c}$$

and interpreted as spatial and time components of tensor quantities over all of Minkowski space $\mathbb{R}^{3+1}$, that is, assuming $A^\mu = (\phi/c, \mathbf{A})$, then the fields $\mathbf{E}$, $\mathbf{B}$, and C are uniquely specified by the following:

$$-\nabla C - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \tag{54a}$$

$$\frac{\partial C}{\partial t} + \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{54b}$$

where $\mu_0\epsilon_0 = 1/c^2$, and where $\mathbf{j}$ is a source current density and ρ is a source charge density defined over all of $\mathbb{R}^{3+1}$.

To prove theorem U2, note that the Maxwell field tensor as defined by

$$F^{\mu\nu}=\begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}, \tag{55}$$

and the definition of the field tensor (55) in terms of the four-vector potential A^μ as defined by

$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu \tag{56}$$

comprise an alternate expression for the relations (53a) and (53b). Equations (55) and (56) are sufficient to demonstrate that $\mathbf{E}$, $\mathbf{B}$, $\mathbf{A}$, and ϕ can be interpreted as components of suitable tensor quantities because the Maxwell field tensor $F^{\mu\nu}$ transforms as a tensor and so by Eq. (56) the four-vector potential A^μ (of electromagnetism) is also a tensor because the right-hand side of Eq. (56) is of the form of a four-curl. Because A^μ can be interpreted as a tensor, its four divergence as defined by

$$C=\partial_\mu A^\mu \tag{57}$$

can also be interpreted as a tensor (of rank 0). Therefore, because Eq. (57) is an alternate expression for Eq. (53c), then C can be interpreted as a tensor quantity as well. By construction, the antisymmetric Maxwell field tensor satisfying Eq. (56) also satisfies the Bianchi identity

$$\partial_\lambda F_{\mu\nu}+\partial_\nu F_{\lambda\mu}+\partial_\mu F_{\nu\lambda}=0, \tag{58}$$

which are Maxwell's source free field equations in tensor form. Equation (58) is also a necessary and sufficient condition that the field tensor $F_{\mu\nu}$ has an auxiliary tensor potential A^μ (and is closed).[23] In three-vector notation Eqs. (3) and (4), when used with the curl of Eq. (53a) and the divergence of Eq. (53b), respectively, accomplish the same task as Eq. (58), and so Maxwell's source free field equations are implied by the definitions (53a) and (53b).

Probably the easiest way to prove the remaining part of theorem U2 is to insert the definitions (53) into Eqs. (54a) and (54b), which reduce to the (uncoupled) wave equations

$$\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu_0\mathbf{j}, \tag{59a}$$

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=-\frac{\rho}{\epsilon_0}. \tag{59b}$$

It is well known that the inhomogeneous and homogeneous solutions of the inhomogeneous hyperbolic wave equations (59) uniquely specify $\mathbf{A}$ and ϕ. Therefore, their first derivatives in space and time, which are assumed to be well defined, are uniquely specified as well, and so by the definitions (53) the fields $\mathbf{E}$, $\mathbf{B}$, and C are uniquely specified and theorem U2 is proved. Note that if Eq. (54a) is substituted into identity (44), and Eq. (54b) is substituted into identity (49), we obtain

$$\mathbf{A}(x)=\int_{V_4'}\mu_0\mathbf{j}(x')G(x,x')d^4x', \tag{60a}$$

$$\frac{\phi(x)}{c}=\frac{1}{c}\int_{V_4'}\frac{\rho(x')}{\epsilon_0}G(x,x')d^4x', \tag{60b}$$

which are inhomogeneous solutions of the wave equations (59) in terms of a suitable Green's function. It was the Green's function solution of Eq. (17b) which led via the delta function identity (23a) to the spatial components (37) and time component (45) of the A^μ identity itself.[6]

Equations (54a) and (54b) reduce to Maxwell's source equations if we set the four-divergence $C=\partial_\mu A^\mu=0$, that is, with the Lorentz gauge condition (more descriptively called a *relativistic transverse gauge* condition).[6,7] It is easily proved that once this relativistic transverse gauge condition is assumed, the resulting Maxwell's equations decouple to give the wave equations (59). The most general Lagrangian density for a (massless) four-vector field A^μ that is no more than quadratic in its variables and their derivatives,[6] and which embodies theorem U2 via the covariant formalism (55)–(57), appears to be

$$\mathcal{L}=-\frac{\epsilon_0c^2}{4}F_{\mu\nu}F^{\mu\nu}-\frac{\lambda\epsilon_0c^2}{2}(\partial_\mu A^\mu)^2+j_\mu A^\mu, \tag{61}$$

where $j^\mu=(\rho c,\mathbf{j})$ and λ is a Lagrange multiplier for the Lorentz constraint term. The covariant Lagrange equation of motion which follows from Eq. (61) is

$$-j^\nu/\epsilon_0c^2=\partial_\mu F^{\mu\nu}+\lambda\partial^\nu(\partial_\mu A^\mu) \tag{62a}$$

$$=\partial_\mu\partial^\mu A^\nu-(1-\lambda)\partial^\nu(\partial_\mu A^\mu). \tag{62b}$$

If we make the physical assumption that all derivatives ∂_μ have equal weight, we may set $\lambda=1$, which reduces Eq. (62b) to

$$\Box A^\nu=\partial_\mu\partial^\mu A^\nu=-j^\nu/\epsilon_0c^2=-\mu_0j^\nu, \tag{63}$$

which is the wave equations (59) in covariant notation.

There is another covariant gauge that I have dubbed the *relativistic longitudinal gauge*,[6,7] where the four-curl $F^{\mu\nu}=0$, which implies via Eq. (55) that $\mathbf{E}=0$ and $\mathbf{B}=0$ as well. With this relativistic longitudinal gauge condition the Lagrange equation of motion (62b) reduces (with $\lambda=1$) to

$$-j^\nu/\epsilon_0c^2=\partial^\nu(\partial_\mu A^\mu)=\partial_\mu(\partial^\nu A^\mu)=\partial_\mu(\partial^\mu A^\nu), \tag{64}$$

where $\partial^\mu A^\nu=\partial^\nu A^\mu$ in this gauge via Eq. (56), which is also the wave equations (59) in covariant notation. Similarly, in three-vector notation the field equations (54) and definitions (53) of theorem U2 for the relativistic longitudinal gauge condition reduce to

$$-\nabla(\nabla\cdot\mathbf{A})-\frac{1}{c^2}\frac{\partial(\nabla\phi)}{\partial t}=-\nabla C=\mu_0\mathbf{j}, \tag{65a}$$

$$\nabla\cdot\frac{\partial\mathbf{A}}{\partial t}+\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=\frac{\partial C}{\partial t}=\frac{\rho}{\epsilon_0}. \tag{65b}$$

The relativistic longitudinal gauge condition $F^{\mu\nu}=0$ implies that

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} = 0, \tag{66a}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = 0, \tag{66b}$$

so that, with the identity (8), it follows that Eqs. (65a) and (65b) reduce to the wave equations (59).

VI. CONCLUDING REMARKS

It can be argued that theorem U2 uniquely specifies the most general three-vector and scalar field (component) equations in Minkowski space for an associated (massless) classical four-vector field A^μ that is assumed to satisfy an inhomogeneous hyperbolic wave equation. That is, the field definitions and equations of theorem U2 follow directly from the use of a Green's function solution technique using an operator delta function four-vector identity. Thus, it should not be surprising that both the relativistic transverse and relativistic longitudinal covariant gauge conditions, when used in conjunction with theorem U2, lead to the same wave equations. I have previously shown that the relativistic transverse and relativistic longitudinal covariant gauge conditions lead respectively to only two classes of classical four-vector fields which are potentially physical.[7] By the latter it is meant that a suitable fully quadratic Lagrangian density (of the type used here but more general in that it has a field mass term and can have complex valued charges) is bounded from below when one or the other of these two covariant gauge conditions is applied.[7]

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[1]F. Rohrlich, "Causality, the Coulomb field, and Newton's law of gravitation," Am. J. Phys. **70**, 411–414 (2002).

[2]J. A. Heras, "Comment on 'Causality, the Coulomb field, and Newton's law of gravitation,' by F. Rohrlich, [Am. J. Phys. **70**, 411–414 (2002)]," Am. J. Phys. **71**, 729–730 (2003).

[3]F. Rohrlich, "The validity of the Helmholtz theorem," Am. J. Phys. **72**, 412–413 (2004).

[4]J. A. Heras, "Jefimenko's formulas with magnetic monopoles and the Liénard-Wiechert fields of a dual-charged particle," Am. J. Phys. **62**, 525–531 (1994).

[5]J. A. Heras, "Comment on 'Alternate 'derivation' of Maxwell's source equations from gauge invariance of classical mechanics,' by James S. Marsh [Am. J. Phys. **61**, 177–178 (1993)]," Am. J. Phys. **62**, 949–950 (1994).

[6]D. A. Woodside, "Uniqueness theorems for classical four-vector fields in Euclidean and Minkowski spaces," J. Math. Phys. **40**, 4911–4943 (1999).

[7]D. A. Woodside, "Classical four-vector fields in the relativistic longitudinal gauge," J. Math. Phys. **41**, 4622–4653 (2000).

[8]A. Sommerfeld, *Mechanics of Deformable Bodies* (Academic, New York, 1964), Vol. II, pp. 147–151.

[9]R. E. Collin, *Field Theory of Guided Waves* (McGraw-Hill, New York, 1960), pp. 564–568.

[10]R. Plonsey and R. E. Collin, *Principles and Applications of Electromagnetic Fields* (McGraw-Hill, New York, 1961), pp. 29–36.

[11]G. Arfken, *Mathematical Methods for Physicists*, 3rd ed. (Academic, Orlando, 1985), pp. 78–84.

[12]R. W. P. King, *Fundamental Electromagnetic Theory*, 2nd ed. (Dover, New York, 1963), p. 163.

[13]A. M. Stewart, "Vector potential of the Coulomb gauge," Eur. J. Phys. **24**, 519–524 (2003).

[14]V. Hnizdo, "Comment on 'Vector potential of the Coulomb gauge'," Eur. J. Phys. **25**, L21–L22 (2004).

[15]V. P. Dmitriyev, "On vector potential of the Coulomb gauge," Eur. J. Phys. **25**, L23–L27 (2004).

[16]A. M. Stewart, "Reply to Comments on 'Vector potential of the Coulomb gauge'," Eur. J. Phys. **25**, L29–L30 (2004).

[17]A. M. Stewart, "On an identity for the volume integral of the square of a vector field," Am. J. Phys. **75**, 561–564 (2007).

[18]J. T. Cushing, *Applied Analytical Mathematics for Physical Scientists* (Wiley, New York, 1975), pp. 511–516.

[19]J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975), pp. 608–612.

[20]N. Cabibbo and E. Ferrari, "Quantum electrodynamics with Dirac monopoles," Nuovo Cimento **23**, 1147–1154 (1962).

[21]O. D. Jefimenko, *Electricity and Magnetism*, 2nd ed. (Electrect Scientific, Star City, 1989), p. 516.

[22]E. Kapuścik, "Generalized Helmholtz theorem and gauge invariance of classical field theories," Lett. Nuovo Cimento Soc. Ital. Fis. **42**, 263–266 (1985).

[23]R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity*, 2nd ed. (McGraw-Hill, New York, 1975), pp. 88, 115–116.