Thus

$$i_{in}(t) = [q/(\tau_1 + \tau_4)]\exp[-t/(\tau_1+\tau_4)]$$
$$= [q/R_s(C_s + C_d)]\exp[-t/R_s(C_s + C_d)],$$

which agrees with other treatments.[9]

Although this current signal case may be helpful in giving a feel for what is happening in one extreme condition, it is not easily attained in practice, as Spieler[10] points out. For an input resistance R_a of 50 Ω (which in other circumstances can often be taken as equivalent to the current signal approximation) one will normally find that the time constant $R_a C_d$ is the predominant one: even for $R_a = 5\ \Omega$ (which could lead to signal termination difficulties) the quantity $R_a C_d$ will still be important. We will thus almost always be forced to use the fuller treatment.

VI. PLASMA EFFECTS

A final complication in semiconductor pulse shapes occurs when heavy ions, such as fission fragments, are being detected. The short range and high ionization associated with these particles leads to the formation of a plasma of ions along the track which is so dense that the collecting electric field cannot immediately penetrate it. Consequently the release of the charge from the plasma takes place over a finite period which is usually characterized by a "plasma decay time," t_p. We have demonstrated elsewhere[11] that by making reasonable assumptions about the form of the decay, one can predict the shape (and particularly the rise) of the corresponding output voltage pulse. Although the rise is now no longer exponential, we have shown that for many practical detectors where $\tau_\alpha \ll t_p \ll \tau_\beta$, the 10% to 90% rise time of the pulse t_r is a valid measure of the plasma time t_p.

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Helmholtz theorem for antisymmetric second-rank tensor fields and electromagnetism with magnetic monopoles

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A generalized Helmholtz's theorem is proved, which states that an antisymmetric second-rank tensor field in 3 + 1 dimensional space-time, which vanishes at spatial infinity, is determined by its divergence and the divergence of its dual. When the divergence of the antisymmetric electromagnetic field strength tensor is equal to the electric charge-current density and the divergence of the dual of the electromagnetic field strength tensor is equal to the magnetic charge-current density, the equations of electromagnetism are obtained. As a convenience, in the solution of the equations of electromagnetism two different four-vector potentials can be used, one of which couples to the electric charge-current density and the other to the magnetic charge-current density.

I. INTRODUCTION

If magnetic monopoles are discovered,[1,2] it is necessary to generalize Maxwell's equations of electromagnetism to include magnetic charge and current density. Although this step has previously been taken,[3] there is a question as to whether the electromagnetic field-strength tensor $F^{\mu\nu}$ is over- or underdetermined[4] by these more general equations. If magnetic monopoles are present, the description of the electromagnetic field-strength tensor in terms of a single nonsingular four-vector potential is no longer adequate.[5]

In this paper a generalization of Helmholtz's theorem[6–8] is proved for antisymmetric second-rank tensors. One way of interpreting Helmholtz's theorem for three-vectors on Euclidean three-space is that it gives an integral formula from which a vector can be constructed if its divergence and curl are known.[9] The divergence of the vector determines its longitudinal part and the curl of the vector determines its transverse part. The generalization of Helmholtz's theorem to antisymmetric second-rank tensors gives a corresponding integral formula from which the tensor can be constructed if its divergence[10] and the divergence of its dual are given. Since the divergence of the dual of an antisymmetric second-rank tensor is a generalization of the curl of a vector, there is a complete correspondence with the theorem for three-vectors.

The theorem that an antisymmetric second-rank tensor is determined by its divergence and the divergence of its dual was stated by Hauser,[11,12] and proved using two different four-vector potentials.[13] The proof given here does not involve potentials, and is proved directly. Potentials can be introduced later as a convenience for solving the two coupled differential equations obtained by specifying the divergence and the divergence of the dual of the antisymmetric second-rank tensor.

When the divergence of the antisymmetric electromagnetic field strength tensor $F^{\mu\nu}$ is specified to be the electric charge-current density four-vector, and the divergence of the dual of the electromagnetic field strength tensor is specified to be the magnetic charge-current density four-vector, the equations of electromagnetism are obtained.[11,14] Maxwell's equations are obtained by setting the magnetic charge-current density four-vector to zero.[11] By taking the divergence of the equations of electromagnetism, it can be shown that both electric and magnetic charge are conserved.[13] The electromagnetic field-strength tensor can be shown from Helmholtz's theorem to be expressible in terms of two different four-potentials.[13] The introduction of the potentials simplifies the solution of the equations of electromagnetism, as it does for Maxwell's equations. In the case of both magnetic and electric charge-current densities, the use of two four-potentials decouples the effect of the magnetic charge-current density from the electric charge-current density in the electromagnetic field-strength tensor.[13]

In Sec. II Helmholtz's theorem for three-vector fields is reviewed and applied to the case of electrostatics with static electric charge density and steady magnetic current density. Helmholtz's theorem is generalized to antisymmetric second-rank tensors in Sec. III. The theorem is applied to the case of electromagnetism in Sec. IV. The conclusions are given in Sec. V.

II. HELMHOLTZ'S THEOREM FOR VECTORS

Helmholtz's theorem for vectors in Euclidean three-dimensional space is reviewed here to establish the notation and set the stage for the later generalization. Once Helmholtz's theorem and its significance are understood in three dimensions, its generalization to $(3+1)$ space-time is straightforward. Helmholtz's theorem is applied here to electrostatics in which both static electric charge density and steady magnetic current density are present.

Helmholtz's theorem can be stated in the form that a vector field $\mathbf{E}(\mathbf{r})$ which vanishes sufficiently rapid at infinity is completely determined by specifying the value of its curl $\nabla\times\mathbf{E}(\mathbf{r})$ and its divergence $\nabla\cdot\mathbf{E}(\mathbf{r})$.[9] The vector field can then be written as

$$\mathbf{E}(\mathbf{r}) = \nabla \int d^3r'\, G(\mathbf{r},\mathbf{r}')\nabla'\cdot\mathbf{E}(\mathbf{r}') - \nabla\times \int d^3r'\, G(\mathbf{r},\mathbf{r}')\nabla'\times\mathbf{E}(\mathbf{r}'), \tag{2.1}$$

which is the sum of a gradient term and a curl term. The function $G(\mathbf{r},\mathbf{r}')$ is the Green's function for the Laplacian

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'). \tag{2.2}$$

If the boundary condition that $G(\mathbf{r},\mathbf{r}')$ vanishes at infinity is used, the solution to Eq. (2.2) is

$$G(\mathbf{r},\mathbf{r}') = -(4\pi|\mathbf{r}-\mathbf{r}'|)^{-1}. \tag{2.3}$$

Equation (2.3) is usually explicitly used in Eq. (2.1). The proof of Eq. (2.1) is given in standard textbooks.[6–8] The significance of Helmholtz's theorem emphasized here is that $\mathbf{E}$ is completely determined by specifying its divergence and curl. Equation (2.1) also shows that $\mathbf{E}$ is decomposed into an irrotational part and a transverse part.

In electrostatics the divergence of the electric field $\mathbf{E}(\mathbf{r})$ is

$$\nabla\cdot\mathbf{E}(\mathbf{r}) = 4\pi\rho_e(\mathbf{r}), \tag{2.4}$$

where $\rho_e(\mathbf{r})$ is the static electric charge density. If magnetic monopoles exist,[1,2] there may be a magnetic current density $\mathbf{J}_m(\mathbf{r})$. In this case the curl of the electric field does not vanish, but is

$$\nabla\times\mathbf{E}(\mathbf{r}) = -(4\pi/c)\mathbf{J}_m(\mathbf{r}). \tag{2.5}$$

Equations (2.4) and (2.5) neither overdetermine nor underdetermine the electric field $\mathbf{E}(\mathbf{r})$, but by Eq. (2.1) uniquely determine it.[4]

For convenience Eq. (2.1) can be written in terms of potentials as

$$\mathbf{E} = -\nabla\phi_e + \nabla\times\mathbf{A}_e. \tag{2.6}$$

The electrostatic scalar potential ϕ_e is

$$\phi_e(\mathbf{r}) = -4\pi\int d^3r'\, G(\mathbf{r},\mathbf{r}')\rho_e(\mathbf{r}'), \tag{2.7}$$

from Eqs. (2.1) and (2.4). The electrostatic vector potential is

$$\mathbf{A}_e(\mathbf{r}) = \left(\frac{4\pi}{c}\right)\int d^3r'\, G(\mathbf{r},\mathbf{r}')\mathbf{J}_m(\mathbf{r}'), \tag{2.8}$$

from Eqs. (2.1) and (2.5). In the usual case of zero magnetic current the electrostatic vector potential is zero. The potentials are not necessary in stating Helmholtz's theorem.

The approach of this section can also be applied to magnetostatics when magnetic monopoles are present. In that case the divergence of the magnetic induction field $\mathbf{B}(\mathbf{r})$ is

$$\nabla\cdot\mathbf{B}(\mathbf{r}) = 4\pi\rho_m(\mathbf{r}), \tag{2.9}$$

where $\rho_m(\mathbf{r})$ is the static magnetic monopole charge density. The curl of the magnetic induction is as usual

$$\nabla\times\mathbf{B}(\mathbf{r}) = (4\pi/c)\mathbf{J}_e(\mathbf{r}), \tag{2.10}$$

where $\mathbf{J}_e(\mathbf{r})$ is the electric current density.

If Eq. (2.1) is applied to $\mathbf{B}$, it is uniquely determined by Eqs. (2.9) and (2.10). The magnetic scalar potential ϕ_m and magnetic vector potential $\mathbf{A}_m$ can be introduced in complete analogy with Eqs. (2.6)–(2.8).

If the electric and magnetic charge densities depend on the time, then Eqs. (2.4) and (2.9) are unchanged. Equation (2.5) becomes in this case

$$\nabla\times\mathbf{E} = -\frac{4\pi}{c}\mathbf{J}_m - \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \tag{2.11}$$

while Eq. (2.10) becomes

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J}_e + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}. \tag{2.12}$$

Concerning Eqs. (2.12) and (2.11) (with $\mathbf{J}_m = 0$), Shadowitz[15] says, "It is seen that they do not satisfy the requirements of Helmholtz's theorem, for the right-hand sides of the two curl equations are unknown, instead of given, functions of space and time. This is too bad insofar as the simplicity of finding the answer is lost, ..." The equations of electromagnetism in Eqs. (2.4), (2.9), (2.11), and (2.12) may be written in terms of an antisymmetric electromagnetic field-strength tensor, which is done in Sec. IV. The generalized Helmholtz's theorem proved in Sec. III for antisymmetric second-rank tensors is just what is required to provide the simplicity of finding solutions to these more general equations of electromagnetism.

III. HELMHOLTZ'S THEOREM FOR AN ANTISYMMETRIC SECOND-RANK TENSOR

The generalization of Helmholtz's theorem to a four-vector in space-time has previously been made by Hauser.[11,12] Assuming that an arbitrary antisymmetric second-rank tensor can be expressed as the curl of a four-vector plus the dual of the curl of another four-vector, he proves as a corollary of his generalized Helmholtz's theorem for four-vectors that a second-rank tensor is determined by its divergence and the divergence of its dual. In this section it is shown that this latter result is the natural statement of Helmholtz's theorem in four-dimensional space-time, and that it can easily be proved directly.

Because the generalization of the curl of a four-vector in space-time is an antisymmetric second-rank tensor, it is easier to formulate Helmholtz's theorem for an arbitrary antisymmetric second-rank tensor field $F^{\mu\nu}(x) = -F^{\nu\mu}(x)$, where $x = x^\mu = (x^0, x^1, x^2, x^3)$ and $x^0 = ct$, than for a four-vector. In Sec. II Helmholtz's theorem was shown to determine a vector when its divergence and curl were specified. Helmholtz's theorem says that an antisymmetric second-rank tensor $F^{\mu\nu}$ is completely specified by giving its divergence $\partial_\mu F^{\mu\nu}$ and the divergence of its dual $\partial_\mu\, {}^*F^{\mu\nu}$, where $\partial^\mu \equiv \partial/\partial x^\mu$ and the Einstein summation convention on repeated Greek indices from 0 to 3 is used. The tensor ${}^*F^{\mu\nu}$ dual to $F^{\mu\nu}$ is defined as

$$ {}^*F^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}, \tag{3.1}$$

where $\epsilon^{\mu\nu\alpha\beta}$ is the completely antisymmetric Levi–Civita tensor with $\epsilon^{0123} = 1$. The contravariant tensor $F^{\mu\nu}$ is transformed to its covariant form by[16]

$$F_{\alpha\beta} = \eta_{\alpha\lambda}\eta_{\beta\sigma}F^{\lambda\sigma}, \tag{3.2}$$

where $\eta_{\mu\nu}$ is the metric tensor of special relativity

$$\eta_{\mu\nu} = \begin{cases} 1 & \text{if } \mu=\nu=0 \\ -1 & \text{if } \mu=\nu=1,2,3 \\ 0 & \text{if } \mu\neq\nu. \end{cases} \tag{3.3}$$

The dual operation in Eq. (3.1) can be applied to any two antisymmetric tensor indices.

The generalized Helmholtz theorem can now be stated.

Theorem. An antisymmetric second-rank tensor field $F^{\mu\nu}(x)$ which vanishes at spatial infinity is completely determined by specifying its divergence $\partial_\mu F^{\mu\nu}$ and the divergence of its dual $\partial_\mu\, {}^*F^{\mu\nu}$. The tensor field $F^{\mu\nu}$ can explicitly be written as

$$F^{\mu\nu}(x) = \int d^4x'\, G^{\mu\nu}{}_\lambda(x,x')\partial'_\alpha F^{\alpha\lambda}(x') - \int d^4x'\, {}^*G^{\mu\nu}{}_\lambda(x,x')\partial'_\alpha\, {}^*F^{\alpha\lambda}(x'), \tag{3.4}$$

where $d^4x' = dx'^0\, dx'^1\, dx'^2\, dx'^3$ is the element of volume in space-time and the integral is taken over all space-time. The third-rank tensor $G^{\mu\nu}{}_\lambda$ is antisymmetric in μ and ν, and is defined as

$$G^{\mu\nu}{}_\lambda(x,x') = (\partial^\mu\delta^\nu{}_\lambda - \partial^\nu\delta^\mu{}_\lambda)G(x,x'), \tag{3.5}$$

where

$$\partial^\mu = \eta^{\mu\alpha}\partial_\alpha = (\partial_0, -\partial_1, -\partial_2, -\partial_3)$$

and $\eta^{\mu\nu}$ is defined by Eq. (3.3). The function $G(x,x')$ is the Green's function for the wave operator $\partial_\mu\partial^\mu = \partial_0^2 - \nabla^2$,

$$\partial_\mu\partial^\mu G(x,x') = \delta(x-x'), \tag{3.6}$$

where $\delta(x-x')$ is the four-dimensional Dirac delta function.

Before proving the theorem, Eq. (3.4) should be compared with Eq. (2.1) for the case of a vector field. The divergence of the second-rank tensor $\partial_\mu F^{\mu\nu}$ is analogous to the divergence of the vector $\nabla\cdot\mathbf{E} = \partial_i E_i$, where the repeated Latin indices are summed from 1 to 3. The tensor $\partial_\mu\, {}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\partial_\mu F_{\alpha\beta}$ is analogous to the curl of a vector, $(\nabla\times\mathbf{E})_i = \epsilon_{ijk}\partial_j E_k$ for $i = 1,2,3$, where ϵ_{ijk} is the antisymmetric Levi–Civita symbol with $\epsilon_{123} = 1$. Thus the integrals on the right-hand side of Eq. (3.4) are analogous to the corresponding integrals in Eq. (2.1). Therefore Eq. (3.4) can be regarded as a natural generalization of Helmholtz's theorem in Eq. (2.1).

The retarded solution of Eq. (3.6) for the Green's function $G(x,x')$ with boundary conditions that it vanishes at spatial infinity is[17]

$$G(x,x') = (4\pi|\mathbf{r}-\mathbf{r}'|)^{-1}\delta(t - t' - |\mathbf{r}-\mathbf{r}'|/c), \tag{3.7}$$

which reduces to Eq. (2.3) [times $-\delta(t-t')$] as $c\to\infty$. The Green's function in Eq. (3.7) has the property that

$$\partial^\mu G(x,x') = -\partial'^\mu G(x,x'), \tag{3.8}$$

which is needed for the proof of Eq. (3.4).

The formula in Eq. (3.4) can be proved by taking its divergence and the divergence of its dual, and showing that an identity results. If the divergence of Eq. (3.4) is taken, it becomes

$$\partial_\mu F^{\mu\nu}(x) = \int d^4x'\, \partial_\mu\partial^\mu G(x,x')\partial'_\alpha F^{\alpha\nu}(x') - \int d^4x'\partial_\mu\partial^\nu G(x,x')\partial'_\alpha F^{\alpha\mu}(x'), \tag{3.9}$$

since

$$\partial_\mu\, {}^*G^{\mu\nu}{}_\lambda = \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}(\partial_\mu\partial_\alpha\eta_{\beta\lambda} - \partial_\mu\partial_\beta\eta_{\alpha\lambda})G = 0 \tag{3.10}$$

because of the antisymmetry of $\epsilon^{\mu\nu\alpha\beta}$ and the symmetry of $\partial_\mu\partial_\alpha$. If Eq. (3.8) is used in the last term on the right-hand side of Eq. (3.9), it can be integrated by parts. The surface contribution at spatial infinity vanishes. The integrand of the remaining term vanishes because it is the double divergence of an antisymmetric tensor. Thus the second integral in Eq. (3.9) vanishes. The first integral reduces to $\partial_\alpha F^{\alpha\nu}$ when Eq. (3.6) is substituted into it and the integration over the delta function is performed, and so an identity is obtained.

If the divergence of the dual of Eq. (3.4) is taken, the result is

$$\partial_\mu\, {}^*F^{\mu\nu}(x) = \int d^4x'\, \partial_\mu\partial^\mu G(x,x')\partial'_\alpha\, {}^*F^{\alpha\nu}(x') - \int d^4x'\, \partial_\mu\partial^\nu G(x,x')\partial'_\alpha\, {}^*F^{\alpha\nu}(x'), \tag{3.11}$$

where Eqs. (3.10) and (3.5) have been used. The dual of the dual of an antisymmetric second-rank tensor is the negative of the tensor. When Eq. (3.8) is used in the second integral in Eq. (3.11) it vanishes after an integration by parts. The first integral in Eq. (3.11) becomes $\partial_\alpha\, {}^*F^{\alpha\nu}$ when Eq. (3.6) is used in it, so Eq. (3.11) reduces to an identity. Therefore the formula in Eq. (3.4) has been established, which proves the theorem.

IV. ELECTROMAGNETISM WITH MAGNETIC MONOPOLES

In order to make the electromagnetic force on a charged particle compatible with the special theory of relativity, the electromagnetic field must be described by a second-rank tensor $F^{\mu\nu}$.[18] If the force is to be a pure force which does not change the particle's rest mass, this second-rank tensor should also be antisymmetric $F^{\mu\nu} = -F^{\nu\mu}$.[19] The antisymmetric electromagnetic field-strength tensor $F^{\mu\nu}$ has the components[16]

$$F^{i0} = E_i \quad (i = 1,2,3)$$
$$F^{ij} = -B_k \tag{4.1}$$

(i,j,k are cyclic permutations of 1,2,3), where $\mathbf{E} = (E_1,E_2,E_3)$ is the electric field and $\mathbf{B} = (B_1,B_2,B_3)$ is the magnetic induction field.

Since the electromagnetic field is an antisymmetric second-rank tensor, the generalized Helmholtz's theorem in Sec. III says that it is completely determined by specifying its divergence and the divergence of its dual. The divergence of the electromagnetic field-strength tensor is a four-vector $j_e^\nu(x)$,

$$\partial_\mu F^{\mu\nu} = (4\pi/c) j_e^\nu, \tag{4.2}$$

and the divergence of its dual is another four-vector $j_m^\nu(x)$,

$$\partial_\mu\, {}^*F^{\mu\nu} = (4\pi/c) j_m^\nu. \tag{4.3}$$

If j_e^ν is taken to be $j_e^\nu = (c\rho_e, \mathbf{J}_e)$, then Eq. (4.2) becomes Eq. (2.4) for $\nu = 0$ with time-dependent electric charge density and Eq. (2.12) for $\nu = 1,2,3$. If j_m^ν is taken to be $j_m^\nu = (c\rho_m, \mathbf{J}_m)$, then Eq. (4.3) becomes Eq. (2.9) for $\nu = 0$ with time-dependent magnetic charge density and Eq. (2.11) for $\nu = 1,2,3$. The space- and time-dependent electric (magnetic) charge density is $\rho_e(\rho_m)$ and the space- and time-dependent electric (magnetic) current density is $\mathbf{J}_e(\mathbf{J}_m)$. In the absence of magnetic charge, $j_m^\nu \equiv 0$, and Eqs. (4.2) and (4.3) reduce to the usual Maxwell's equations. The generalized Helmholtz's theorem in Sec. III shows that Eqs. (4.2) and (4.3) neither overdetermine nor underdetermine the electromagnetic field strength tensor $F^{\mu\nu}$, but completely specify it.[4]

If the divergence of Eqs. (4.2) and (4.3) is taken, the equations of continuity

$$\partial_\nu j_i^\nu = 0, \quad i = e,m \tag{4.4}$$

are obtained. Equation (4.4) states that both electric and magnetic charge are conserved.

It is not necessary to introduce potentials, but they can be introduced for convenience. Equation (3.4) shows that when magnetic charge is present, two independent four-vector potentials, A_e^μ and A_m^μ, are required to specify $F^{\mu\nu}$. If Eqs. (4.2) and (4.3) are substituted into Eq. (3.4), it can be written as

$$F^{\mu\nu} = (\partial^\mu A_e^\nu - \partial^\nu A_e^\mu) - {}^*(\partial^\mu A_m^\nu - \partial^\nu A_m^\mu), \tag{4.5}$$

a form that was first used by Cabibbo and Ferrari.[13] This form is the one assumed by Hauser[11,12] to prove his corollary to Helmholtz's theorem. It emerges here in a natural way as a corollary to the generalized Helmholtz's theorem of Sec. III, which shows that all antisymmetric second-rank tensors can be written in this form. The four-potential A_i^μ is defined as

$$A_i^\mu(x) = \frac{4\pi}{c}\int d^4x'\, G(x,x') j_i^\mu(x'), \quad i = e,m, \tag{4.6}$$

in terms of the charge-current density four-vector $j_i^\mu(x)$ for $i = e,m$.

When Eq. (4.5) is substituted into Eqs. (4.2) and (4.3) the equations

$$\partial_\mu \partial^\mu A_i^\nu - \partial^\nu(\partial_\mu A_i^\mu) = (4\pi/c) j_i^\nu, \quad i = e,m, \tag{4.7}$$

are obtained. If the Lorentz gauge conditions $\partial_\mu A_i^\mu = 0$ for $i = e,m$, are used, the solution of Eq. (4.7) is given by Eq. (4.6). By introducing the potentials A_i^μ for $i = e,m$, the coupled differential equations in Eqs. (4.2) and (4.3) result in Eq. (4.7) for $i = e,m$, which are decoupled and easy to solve.[15]

In general gauge transformations[13] can be made on A_i^μ,

$$A_i'^\mu = A_i^\mu - \partial^\mu \Lambda_i, \quad i = e,m, \tag{4.8}$$

where $\Lambda_e(x)$ and $\Lambda_m(x)$ are two independent functions of x. The electromagnetic field strength tensor in Eq. (4.5) is invariant under the transformations in Eq. (4.8).

V. CONCLUSION

For a three-vector field which vanishes at infinity, Helmholtz's theorem[6–8] says that it is uniquely determined by specifying its divergence and curl.[9] The curl of a four-vector is not a four-vector, but an antisymmetric second-rank tensor. Therefore, the natural extension of Helmholtz's theorem is to antisymmetric second-rank tensors in space-time. The divergence of a second-rank tensor, which is a four-vector, is defined in analogy with the divergence of a three-vector. Likewise, the divergence of the dual of an antisymmetric second-rank tensor, which is also a four-vector, is analogous to the curl of a three-vector. The generalized Helmholtz's theorem then states that an antisymmetric second-rank tensor which vanishes at spatial infinity is completely determined by specifying its divergence and the divergence of its dual.

The generalized Helmholtz's theorem is applied to electromagnetism, which is described by an antisymmetric second-rank electromagnetic field-strength tensor. The divergence of the electromagnetic field-strength tensor is set equal to the electric charge-current density four-vector (times $4\pi/c$). The divergence of the dual of the electromagnetic field-strength tensor is set equal to the magnetic charge-current density four-vector (times $4\pi/c$). These two equations are a complete description of electromagnetism, because the generalized Helmholtz's theorem states that the electromagnetic field-strength tensor is completely determined. The conservation of electric and magnetic charge follows from the antisymmetry of the electromagnetic field-strength tensor. If the magnetic charge-current density four-vector is identically zero, Maxwell's equations are obtained.

The introduction of potentials is not necessary, but is a convenience in solving the equations of electromagnetism.

Unlike Maxwell's equations, where one four-vector potential is sufficient, the equations of electromagnetism with magnetic charge-current density require the use of two different four-vector potentials.[13] The expression of an antisymmetric second-rank tensor in terms of two different four-vector potentials was previously postulated,[11–13] but finds its natural justification in the generalized Helmholtz's theorem. The use of potentials decouples the equations of electromagnetism. One four-vector potential is determined by the electric charge-current density four-vector, while the other one is determined by the magnetic charge-current density four-vector.

Once electromagnetism is shown to be described by an antisymmetric second-rank tensor, as it can be from the Lorentz force,[18,19] the equations of electromagnetism are forced on us because of the generalized Helmholtz's theorem. The theorem shows that the electromagnetic field-strength tensor is then neither overdetermined nor underdetermined,[4] but completely specified.

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Electrostatic images by multipole expansion

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We propose a simple method for solving some electrostatic problems normally treated by images. The method is based on the multipole expansion of the electrostatic potential due to both real and image charges. The magnitude and the location of the image charges are found by imposing the proper boundary conditions. Some typical problems with spherical and with spheroidal symmetries are worked out explicitly.

I. INTRODUCTION

It is well known that, in the absence of space charge, the electrostatic potential ψ satisfies the Laplace equation[1]

$$\nabla^2\psi = 0. \tag{1}$$

Also,[2]

$$\psi(\bar{r}) = \int_s \frac{\rho(\bar{r}')da'}{|\bar{r} - \bar{r}|} + \int_v \frac{\rho(\bar{r}')dv'}{|\bar{r} - \bar{r}'|}, \tag{2}$$

where σ and ρ are the surface and volumetric charge densities, respectively.

Once the potential is found, one can calculate the electric field $\bar{E}$ from the relation

$$\bar{E} = -\nabla\psi.$$

In some problems, however, σ and ρ are unknown. Thus Eq. (2) cannot be used, but it is still possible to solve Eq. (1). For instance, suppose that we have a certain number of charged conductors in a vacuum, for which the potential or total charge is given; since the surface charge distributions are unknown it is necessary to find a solution of Eq. (1) consistent with the specified potentials or charges on the conductors.

Alternatively, we can solve the above problem by replacing the unknown surface charge distributions on the conductors by a set of discrete charges that fit the same boundary conditions for the potential. This "method of images" is based on the uniqueness of the solution of Eq. (1). It allows one to find the potential, the electric field, and the charge distributions without solving explicitly this equation.[3]

When dielectric media are present, the method is formally similar. In this case one applies the continuity of the tangential component of the electric field strength $\bar{E}$ and the normal component of the electric flux density $\bar{D}$, across the boundary separating the media.[4]

The method of images is discussed in textbooks on electromagnetic theory.[1–4] Unfortunately, they do not mention