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where

$$B(i\bar{w}_B) = L\left(\frac{i\bar{u} + i\bar{v} - \bar{u}\times\bar{v}}{1+\bar{u}\cdot\bar{v}}\right)R\left(\frac{\bar{u}\times\bar{v}}{1+\bar{u}\cdot\bar{v}}\right). \quad (13)$$

Applying the combination formula (6) to this equation yields

$$\bar{w}_B = [\bar{u}(1 + 2\bar{u}\cdot\bar{v} + \bar{v}^2) + \bar{v}(1 - \bar{u}^2)]/ (1 + 2\bar{u}\cdot\bar{v} + \bar{u}^2\bar{v}^2). \quad (14)$$

Note that $\bar{w}_B$ is perpendicular to $\bar{w}_R$. Computing the magnitude of $\bar{w}_B$ gives the rapidity of the resultant boost, which may be written as

$$\cosh^2\left(\frac{\xi w_B}{2}\right) = \left(\cosh\frac{\xi_u}{2}\cosh\frac{\xi_v}{2}\hat{u} + \sinh\frac{\xi_u}{2}\sinh\frac{\xi_v}{2}\hat{v}\right)^2. \quad (15)$$

This is, of course, a generalization of the addition formula for hyperbolic cosines. The rotation which occurs here is called the Wigner little group rotation in the group-theoretical analysis of the irreducible representations of the Poincaré group.[5]

These formulas simplify in the ultrarelativistic limit $\bar{u}^2 \approx \bar{v}^2 \approx 1$. In this case Eqs. (11) and (14) reduce to $\cos\theta_R \approx \bar{u}\cdot\bar{v}, \bar{w}_B \approx \bar{u}$, where θ_R is the angle of the Wigner rotation. Equation (12) becomes

$$B(i\bar{u})B(i\bar{v}) \approx B(i\bar{u})R\{-[(\bar{u}\times\bar{v})/|\bar{u}\times\bar{v}|]\tan(\theta_R/2)\}. \quad (16)$$

The boost $B(i\bar{u})$ carries an arbitrary coordinate system into a coordinate system traveling along $\bar{u}$ with near-light velocity without rotating its axes. If the boost $B(i\bar{u})$ is preceded by a boost $B(i\bar{v})$ we again end up with a coordinate system traveling along $\bar{u}$ with near-light velocity, but the axes are now rotated with respect to those of the original coordinate system by a rotation equal to that which rotates the vector $\bar{v}$ into the vector $\bar{u}$.

As a closing remark, we note that the formula (6) exhibits the fact that a vector addition law results only for infinitesimal transformations. The cross-product term, which expresses the noncommutativity, is related to the structure constants ϵ_{ijk} of the rotation group: $(\bar{u}\times\bar{v})_i = \Sigma_{j,k}\,\epsilon_{ijk}u_jv_k$. The structure constants of a general group express the deviation of the commutator of two group operations from the identity transformation.[6] For infinitesimal rotations our formula for the commutator yields

$$R^{-1}(\bar{u})R^{-1}(\bar{v})R(\bar{u})R(\bar{v}) \approx R\,[2(\bar{u}\times\bar{v})]. \quad (17)$$

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Helmholtz's theorem revisited

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Helmholtz's theorem for three-vectors is applicable to electrostatics and magnetostatics, but it must be generalized to antisymmetric second-rank tensors to be applicable to electromagnetism. A proof of Helmholtz's theorem for three-vectors, which is shorter than the usual one, is given. Helmholtz's theorem is shown to be a special case of the Hodge decomposition theorem in the theory of differential forms.

The purpose of this article is to comment on recent papers[1–3] in this Journal dealing with Helmholtz's theorem. Helmholtz's theorem for three-vectors is applicable to electrostatics and magnetostatics.[4] For electromagnetism it is necessary to generalize Helmholtz's theorem to antisymmetric second-rank tensors.[2] A proof of Helmholtz's theorem for three-vectors, which is shorter than the usual one, is given. Helmholtz's theorem is shown to be a special case of the Hodge decomposition theorem for differential forms.[5]

Miller[1] has recently discussed Helmholtz's theorem in classical electromagnetism. His remark about sources in Maxwell's equations is, however, correct only for electrostatics and magnetostatics. Miller states, "The significance of this [Helmholtz's theorem] to Maxwell's equations of the electromagnetic field is evident since each of the four commonly stated Maxwell equations is in terms of a divergence or a curl of one of the electromagnetic field vectors, and, hence, each specifies a source relation... ." Shadowitz[4] recognized that this statement is incorrect. In the Ampere–Maxwell law the curl of the magnetic field strength **H** is equal to the current density **J** plus the so-called displacement current $\partial\mathbf{D}/\partial t$. As Rosser[6] has pointed out, the displacement current for vacuum is not, however, a source of the electromagnetic field. Neither is the negative time rate of change of the magnetic induction **B** a source term in

Faraday's law. The only sources in Maxwell's equations are the charge density ρ and the current density $\mathbf{J}$. In order to apply Helmholtz's theorem to electromagnetism, it is necessary to generalize it to antisymmetric second-rank tensor fields, as was done recently.[2]

A proof of Helmholtz's theorem for three-vectors, which is shorter and slightly more general than the usual one,[7,8] is given here. Helmholtz's theorem states that a vector field $\mathbf{Z}(\mathbf{r})$ which vanishes at the boundaries may be written as the sum of two terms, one of which is irrotational and the other solenoidal. Consider the Laplacian of a vector field $\mathbf{V}(\mathbf{r})$[9]:

$$-\nabla^2\mathbf{V} = -\nabla(\nabla\cdot\mathbf{V}) + \nabla\times\nabla\times\mathbf{V}. \quad (1)$$

If the vector field $\mathbf{Z}$ is taken to be

$$\nabla^2\mathbf{V} = -\mathbf{Z}, \quad (2)$$

then it follows from Eq. (1) that

$$\mathbf{Z} = -\nabla U + \nabla\times\mathbf{W}. \quad (3)$$

The scalar field U is

$$U = \nabla\cdot\mathbf{V} \quad (4)$$

and the vector field $\mathbf{W}$ is

$$\mathbf{W} = \nabla\times\mathbf{V}. \quad (5)$$

Equation (3) is Helmholtz's theorem, where $-\nabla U$ is irrotational and $\nabla\times\mathbf{W}$ solenoidal.

A corollary of Helmholtz's theorem is that the vector field $\mathbf{Z}(\mathbf{r})$ which vanishes at the boundaries is determined by its divergence and curl. The uniqueness is proved by Arfken.[7] From the above equations this corollary is also easily proved.[10] The divergence of Eq. (3) is Poisson's equation

$$\nabla^2 U = -\nabla\cdot\mathbf{Z}. \quad (6)$$

The solution to Poisson's equation for a function which vanishes at the boundaries of the volume V is

$$U(\mathbf{r}) = \int_V d^3r'\, G(\mathbf{r},\mathbf{r}')\ \nabla'\cdot\mathbf{Z}(\mathbf{r}'). \quad (7)$$

The Green's function is

$$G(\mathbf{r},\mathbf{r}') = G_1(\mathbf{r},\mathbf{r}') + G_0(\mathbf{r},\mathbf{r}'), \quad (8)$$

where $G_1(\mathbf{r},\mathbf{r}') = (4\pi|\mathbf{r}-\mathbf{r}'|)^{-1}$ and $G_0(\mathbf{r},\mathbf{r}')$ is a solution to Laplace's equation as determined from the boundary condition that $G=0$ on the boundary of V.[11] The curl of Eq. (3) gives

$$\nabla\times\mathbf{Z} = \nabla\times\nabla\times\mathbf{W} = \nabla(\nabla\cdot\mathbf{W}) - \nabla^2\mathbf{W} \quad (9)$$

from Eq. (1). Because of Eq. (5), $\nabla\cdot\mathbf{W}=0$, and Eq. (9) is Poisson's equation. The solution to the equation is

$$\mathbf{W}(\mathbf{r}) = \int_V d^3\mathbf{r}'\, G(\mathbf{r},\mathbf{r}')\ \nabla'\times\mathbf{Z}(\mathbf{r}'). \quad (10)$$

Therefore $\mathbf{Z}$ is determined by its divergence and curl when Eqs. (7) and (10) are used in Eq. (3). If only the Green's function G_1 in Eq. (8) is used in Eqs. (7) and (10), then Helmholtz's theorem in Eq. (3) should have a harmonic term $\mathbf{Z}_0$ added to it which satisfies Laplace's equation in order to satisfy the boundary conditions.

To apply Helmholtz's theorem to electromagnetism it must be extended to an antisymmetric second-rank tensor field.[2] An antisymmetric second-rank tensor $F^{\mu\nu}$ can be expressed as[12–14]

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) - {}^*(\partial_\mu C_\nu - \partial_\nu C_\mu) + F^{(0)}_{\mu\nu}, \quad (11)$$

where $A_\mu(x)$ and $C_\mu(x)$ are nonsingular four-vectors. The dual, denoted by an asterisk, of an antisymmetric second-rank tensor $F_{\mu\nu}$ is defined as

$${}^*F_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}, \quad (12)$$

where $\varepsilon_{\mu\nu\alpha\beta}$ is the totally antisymmetric Levi-Civita tensor with $\varepsilon_{0123} = -1$. The first and second terms on the right-hand side of Eq. (11) are analogous to the corresponding terms in Eq. (3). The third term $F^{(0)}_{\mu\nu}$ on the right-hand side of Eq. (11) is a solution to the homogeneous wave equation, which is added to satisfy the boundary conditions.[15] It is analogous to a solution of Laplace's equation, which can be added to Eq. (3) to satisfy the boundary conditions.

If $F_{\mu\nu}$ is the electromagnetic field strength tensor, A_μ is the usual nonsingular four-vector potential which couples to the electric charge–current density. If there are magnetic monopoles, then C_μ is another independent nonsingular four-vector potential which couples to the *magnetic* charge–current density. There is no difficulty in including magnetic monopoles in electromagnetism,[2] contrary to Schleifer, who says, "It is within this language [differential forms] that the question of the apparent nonexistence of magnetic monopoles is most compelling."[16] Magnetic monopoles may still be treated without the four-potential C_μ. In the Dirac string theory,[17] A_μ has a line singularity which is not observable. The Wu and Yang method[18] of treating magnetic monopoles may also be used. In this method two different four-potentials A_μ and A'_μ, related to each other by a gauge transformation, are used in different regions of space. Nevertheless, if magnetic monopoles do not exist, $C_\mu = 0$ and A_μ is nonsingular. Equation (11) reduces to the usual expression for the electromagnetic field strength tensor in Maxwell's theory.

The antisymmetric second-rank tensor $F_{\mu\nu}$ may be written in a compact coordinate-independent way by using the language of differential forms.[3,5] The content of the equations does not change, so that the above statement of Schleifer[16] is still invalid. Schleifer observes that Helmholtz's theorem for three-vectors is equivalent to the Hodge decomposition theorem of differential forms in three-space.[3] The generalization of Helmholtz's theorem to antisymmetric second-rank tensors in Eq. (11) is the Hodge decomposition theorem for two-forms. If Eq. (11) is multiplied by one-half the two-form $dx^\mu\wedge dx^\nu = -dx^\nu\wedge dx^\mu$, the exterior product of dx^μ and dx^ν,[19] the result is

$$F = dA - {*dC} + F^{(0)}, \quad (13)$$

where the asterisk denotes the Hodge star operator.[20] The electromagnetic field strength two-form is

$$F = \tfrac{1}{2}F_{\mu\nu}dx^\mu\wedge dx^\nu, \quad (14)$$

where the repeated Greek indices are summed from 0–3. The two-form dA is defined as

$$dA = \partial_\mu A_\nu\ dx^\mu\wedge dx^\nu, \quad (15)$$

where $A = A_\nu dx^\nu$ is a one-form. The two-form $*dC$ is[21]

$$*dC = \tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\partial^\alpha C^\beta\ dx^\mu\wedge dx^\nu, \quad (16)$$

where $C = C_\nu dx^\nu$ is a one-form and $\varepsilon_{\mu\nu\alpha\beta}$ is the Levi-Civita tensor. If we define a three-form B such that $C = *B$, then Eq. (13) can be written as

$$F = dA + \delta B + F^{(0)}, \tag{17}$$

where the codifferential[21] δ of B is defined as $\delta B = -*d*B$ and the two-form $F^{(0)}$ is a solution to the wave equation. Equation (17) is a special case of the Hodge decomposition theorem.[22–24] The Hodge decomposition theorem is the natural generalization of Helmholtz's theorem to an arbitrary dimensional space.

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Cooling by immersion in liquid nitrogen

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When an object is cooled by immersion in a liquid, there is an unexpected increase in the violence of boiling just before the boiling stops. Most people seem fascinated by this phenomenon yet few are acquainted with its explanation in terms of a change in the heat-transfer mechanism from film boiling to nucleate boiling. We have developed two variations of an intermediate level undergraduate laboratory experiment to measure the heat-transfer rate after a sample is immersed in liquid nitrogen. The temperature of the sample, as measured by a thermocouple, is recorded as a function of time using either a potentiometer strip-chart recorder or a digital voltmeter–microcomputer combination. The heat-transfer rate as a function of sample temperature is computed from these results, and the reason for the effect is clearly seen.

I. INTRODUCTION

When a metal object such as a steel bolt is suddenly immersed in liquid nitrogen, the liquid starts boiling violently. As time goes on, the violence of the boiling slowly decreases, and soon it appears as if the boiling is just about to stop. But instead of stopping, there is suddenly a big increase in the violence of boiling and then the boiling stops.

If you have never seen this effect for yourself, you may question whether it really occurs. You can easily observe it if liquid nitrogen is available and may already have observed it when filling a room temperature vacuum system cold trap with liquid nitrogen. If liquid nitrogen is not available, you can see the same effect by dropping an incandescent piece of metal into water.

The explanation for this effect, called the Leidenfrost effect,[1] lies in the nature of heat transfer between a solid surface and a colder surrounding liquid.[2] This apparently simple process consists of six distinct and identifiable regimes of pool boiling.

Figure 1 shows the principal boiling regimes of water at atmospheric pressure as reported by Farber and Scorah.[3] In their experiment, an electrically heated platinum wire was immersed in water, and the power input $\dot{Q}$ required to maintain the wire at various temperatures T above the saturation temperature of the water T_{SAT} was measured.