How the potentials in different gauges yield the same retarded electric and magnetic fields

José A. Heras
Departamento de Física, E. S. F. M., Instituto Politécnico Nacional, México D. F. México
and Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803-4001

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This paper presents a simple and systematic method for showing how the potentials in the Lorentz, Coulomb, Kirchhoff, velocity, and temporal gauges yield the same retarded electric and magnetic fields. The method uses the appropriate dynamical equations for the scalar and vector potentials to obtain two wave equations whose retarded solutions lead to the electric and magnetic fields. The advantage of this method is that it does not use explicit expressions for the potentials in the various gauges, which are generally simple to obtain for the scalar potential but difficult to calculate for the vector potential. The spurious character of the term generated by the scalar potential in the Coulomb, Kirchhoff, and velocity gauges is noted. The nonspurious character of the term generated by the scalar potential in the Lorenz gauge is emphasized.

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I. INTRODUCTION

As is well known, the advantage of the Coulomb gauge is that the scalar potential in this gauge is simple to obtain, but the disadvantage is that the vector potential in this gauge is difficult to calculate. This characteristic of the Coulomb gauge is why the explicit demonstration that the potentials in this gauge yield the retarded electric and magnetic fields is not usually presented in textbooks. Some textbooks mention a paper by Brill and Goodman in which an elaborate proof that the potentials in the Coulomb and Lorenz gauges yield the same retarded electric and magnetic fields is presented; this proof is restricted to sources with harmonic time dependence.

It is conceptually important to emphasize the fact that the causal behavior of the retarded electric and magnetic fields is not lost when they are expressed in terms of potentials in the Coulomb gauge, despite the result that the scalar potential in this gauge propagates instantaneously, which suggests a lost of causality in the electric field. Proofs that the potentials in other gauges such as the temporal or velocity gauges yield the retarded electric and magnetic fields are omitted in textbooks. This omission is reasonable because these gauges are not usually mentioned in textbooks of electrodynamics. The velocity gauge is one in which the scalar potential propagates with an arbitrary velocity and the temporal gauge is one in which the scalar potential is identically zero.

In a recent paper Jackson and Okun reviewed the history that led to the conclusion that potentials in different gauges describe the same physical fields. In a subsequent paper, Jackson derived novel expressions for the vector potential in the Coulomb, velocity, and temporal gauges and demonstrated how these expressions for the vector potential together with the associated expressions for the scalar potential lead to the same retarded electric and magnetic fields. Jackson emphasized that “...whatever propagation or nonpropagation characteristics are exhibited by the potentials in a particular gauge, the electric and magnetic fields are always the same and display the experimentally verified properties of causality and propagation at the speed of light.” Rohrlich recently discussed causality in the Coulomb gauge. The present author has used two methods to show that the Coulomb gauge potentials yield the retarded electric field and rediscovers the Kirchhoff gauge in which the scalar potential “propagates” with the imaginary speed iε, where ε is the speed of light. In a recent paper, Yang discussed the velocity gauge.

To show that potentials in different gauges yield the same retarded fields, we usually first derive explicit expressions for the scalar and vector potentials in a particular gauge. The retarded fields are then obtained by differentiation of the potentials. The practical difficulty of this approach is that the derivation of explicit expressions for potentials in most gauges is not simple, particularly for the vector potential. The question arises: Is it necessary to have explicit expressions for the potentials in a particular gauge to show that they lead to the retarded electric and magnetic fields? The answer is negative, at least for the gauges considered in this paper.

In this paper we present a simple and systematic method to show how the potentials in the Lorentz, Coulomb, Kirchhoff, velocity, and temporal gauges yield the same retarded electric and magnetic fields. Instead of using explicit expressions for the scalar and vector potentials in these gauges, we use the appropriate dynamical equations of the potentials to obtain two wave equations, whose retarded solutions lead to the retarded fields.

In Sec. II we define the Lorentz, Coulomb, Kirchhoff, velocity, and temporal gauges. In Sec. III we review the usual proof that the Lorentz gauge potentials lead to the retarded fields and use the alternative method to show how these potentials yield the retarded fields. In Sec. IV we apply the same method to the Coulomb gauge potentials. In Sec. V we discuss the steps of the proposed method. In Sec. VI we apply the method to the Kirchhoff gauge potentials, and in Secs. VII and VIII we apply the method to the velocity gauge potentials and the temporal gauge vector potential, respectively. In Sec. IX we emphasize the spurious character of the gradient of the scalar potential in the Coulomb, Kirchhoff, and velocity gauges and the nonspurious character of the gradient of the scalar potential in the Lorenz gauge. We suggest that the Lorenz gauge potentials may be interpreted as physical quantities. In Sec. X we present some concluding remarks.
II. ELECTROMAGNETIC GAUGES

It is well known that the electric and magnetic fields \( \mathbf{E} \) and \( \mathbf{B} \) are determined from the scalar and vector potentials \( \Phi \) and \( \mathbf{A} \) according to

\[
\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t},
\]

\[
\mathbf{B} = \nabla \times \mathbf{A}.
\]

We use SI units and consider fields with localized sources in vacuum. The fields \( \mathbf{E} \) and \( \mathbf{B} \) are invariant under the gauge transformations

\[
\Phi' = \Phi - \frac{\partial \chi}{\partial t},
\]

\[
\mathbf{A}' = \mathbf{A} + \nabla \chi,
\]

where \( \chi \) is an arbitrary time-dependent gauge function. The inhomogeneous Maxwell equations together with Eq. (1) lead to the coupled equations

\[
\nabla^2 \Phi = -\frac{1}{\varepsilon_0} \rho - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}),
\]

\[
\frac{\Box^2}{c^2} \mathbf{A} = -\mu_0 \mathbf{J} + \nabla \left( \nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right),
\]

where \( \Box^2 = \nabla^2 - (1/c^2) \partial^2/\partial t^2 \) is the D’Alambertian operator and \( \rho \) and \( \mathbf{J} \) are the charge and current densities, respectively. The arbitrariness of the gauge function \( \chi \) in Eq. (2) allows a gauge condition to be chosen. We will consider five gauge conditions,

\[
\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0, \quad \text{Lorenz gauge,}^{1,2}
\]

\[
\nabla \cdot \mathbf{A} = 0, \quad \text{Coulomb gauge,}^{3,5}
\]

\[
\nabla \cdot \mathbf{A} - \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0, \quad \text{Kirchhoff gauge,}^{9}
\]

\[
\nabla \cdot \mathbf{A} + \frac{1}{v^2} \frac{\partial \Phi}{\partial t} = 0, \quad \text{Velocity gauge,}^{5,11,12}
\]

\[
\Phi = 0, \quad \text{Temporal gauge.}^{5}
\]

We note that the velocity gauge contains the Lorenz gauge \((v=c)\), the Coulomb gauge \((v=\infty)\), and the Kirchhoff gauge \((v=ic)\).

III. LORENZ GAUGE

The most popular gauge is the Lorenz gauge, which allows us to uncouple Eq. (3) so that the scalar and vector potentials are described by symmetrical (uncoupled) equations, which is a characteristic of this gauge. If we assume the Lorenz gauge (4), then Eq. (3) becomes

\[
\Box^2 \Phi_L = -\frac{1}{\varepsilon_0} \rho,
\]

\[
\Box^2 \mathbf{A}_L = -\mu_0 \mathbf{J}.
\]

The notation \( \Phi_L \) and \( \mathbf{A}_L \) indicates that these potentials are in the Lorenz gauge. An advantage of the Lorenz gauge is that it can be written in the relativistically covariant form

\[
\Phi_L(x,t) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \left( \frac{1}{R} \rho(x', t-R/c) \right),
\]

\[
\mathbf{A}_L(x,t) = \frac{\mu_0}{4\pi} \int d^3x' \left( \frac{1}{R} \mathbf{J}(x', t-R/c) \right),
\]

where \( R \) is the magnitude of the vector \( \mathbf{R}=x-x' \) with \( x \) the field point and \( x' \) the source point. The integrals in Eq. (10) are over all space. An advantage of the Lorenz gauge is that it can be written in the relativistically covariant form \( \partial_\mu A_\mu = 0 \), where \( \partial_\mu = ((1/c)\partial/\partial t, \nabla) \) and \( A_\mu = (\Phi/c, \mathbf{A}) \). Greek indices run from 0 to 3; the signature of the Minkowski metric is \((1,-1,-1,-1)\) and summation on repeated indices is understood. Equations (9) and (10) can also be written in a relativistically covariant form.

Another characteristic of the Lorenz gauge is that the scalar potential yields a retarded term that can be written as

\[
-\nabla \Phi_L(x,t) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \left( \hat{\mathbf{R}} \frac{R}{R^2} \rho(x', t-R/c) \right)
\]

\[
+ \hat{\mathbf{R}} \frac{R}{R^2} \frac{\partial \rho(x', t-R/c)}{\partial (t-R/c)},
\]

where \( \hat{\mathbf{R}}=R/R \). This form displays the properties of causality and propagation at the speed of light. We anticipate that the gradient of the scalar potential in the other gauges considered in this paper does not satisfy the above properties.

Given the potentials \( \Phi_L \) and \( \mathbf{A}_L \), the electric and magnetic fields can be derived by the usual prescription

\[
\mathbf{E} = -\nabla \Phi_L - \frac{\partial \mathbf{A}_L}{\partial t} = -\nabla \left( \frac{1}{4\pi\varepsilon_0} \int d^3x' \frac{1}{R}[\rho] \right)
\]

\[
- \frac{\partial}{\partial t} \left( \frac{\mu_0}{4\pi} \int d^3x' \frac{1}{R}[\mathbf{J}] \right),
\]

\[
\mathbf{B} = \nabla \times \mathbf{A}_L = \nabla \times \left( \frac{\mu_0}{4\pi} \int d^3x' \frac{1}{R}[\mathbf{J}] \right),
\]

where we have introduced the retarded symbol \( [ \ ] \) to indicate that the enclosed quantity is to be evaluated at the retarded time \( t'=t-R/c \).

After an integration by parts, Eq. (12) becomes the usual electric and magnetic fields

\[
\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \int d^3x' \frac{1}{R} \left[ -\nabla' \rho - \frac{1}{c^2} \frac{\partial \mathbf{J}}{\partial t'} \right],
\]

\[
\mathbf{B} = \frac{\mu_0}{4\pi} \int d^3x' \frac{1}{R} [\nabla' \times \mathbf{J}],
\]
Equation (13) can also be obtained without considering Eq. (10) by first taking the negative of the gradient of Eq. (9a) and the negative of the time derivative of Eq. (9b),

\[-\nabla^2 \Phi_L = \frac{1}{\varepsilon_0} \nabla \rho, \quad (14a)\]

\[-\frac{\partial}{\partial t} \nabla \Phi_L = \frac{\mu_0}{4\pi} \frac{\partial}{\partial t} \nabla \Phi_L = \frac{\mu_0}{4\pi} \int d^3\chi' \frac{1}{R} \left[ -\nabla' \rho + \frac{1}{c^2} \frac{\partial J}{\partial t'} \right]. \quad (14b)\]

The retarded solutions of Eq. (14) are given by

\[-\nabla \Phi_L = \frac{1}{4\pi\varepsilon_0} \int d^3\chi' \frac{1}{R} \left[ -\nabla' \rho - \frac{1}{c^2} \frac{\partial J}{\partial t'} \right], \quad (15a)\]

\[-\frac{\partial}{\partial t} \nabla \Phi_L = \frac{\mu_0}{4\pi} \int d^3\chi' \frac{1}{R} \left[ -\frac{\partial J}{\partial t'} \right]. \quad (15b)\]

Alternatively, the curl of Eq. (9b) gives

\[\square^2 (\nabla \times A_L) = -\mu_0 \nabla \times J, \quad (16)\]

with the retarded solution

\[\nabla \times A_L = \frac{\mu_0}{4\pi} \int d^3\chi' \frac{1}{R} \left[ \nabla' \times J \right]. \quad (17)\]

Equations (15) and (17) yield the usual form of the retarded electric and magnetic fields

\[E = -\nabla \Phi_L - \frac{\partial A_L}{\partial t} = \frac{1}{4\pi\varepsilon_0} \int d^3\chi' \frac{1}{R} \left[ -\nabla' \rho - \frac{1}{c^2} \frac{\partial J}{\partial t'} \right], \quad (18a)\]

\[B = \nabla \times A_L = \frac{\mu_0}{4\pi} \int d^3\chi' \frac{1}{R} \left[ \nabla' \times J \right]. \quad (18b)\]

This alternative method of obtaining the electric and magnetic fields, which works directly with Eq. (9), does not have any practical advantage for the Lorenz gauge compared to the traditional method that uses Eq. (10). We will see in the next sections that the alternative method is advantageous when applied to potentials in other gauges.

IV. COULOMB GAUGE

A less popular gauge in textbooks is the Coulomb gauge. In this gauge the scalar potential satisfies an instantaneous Poisson equation, which is a peculiar characteristic of this gauge. If we assume the Coulomb gauge in Eq. (5), then Eqs. (3) become the coupled equations:

\[\nabla^2 \Phi_C = -\frac{1}{\varepsilon_0} \rho, \quad (19a)\]

\[\square^2 A_C = -\mu_0 J + \frac{1}{c^2} \frac{\partial \Phi_C}{\partial t}, \quad (19b)\]

where we have used the notation \(\Phi_C\) and \(A_C\) to specify that these potentials are in the Coulomb gauge. As pointed out in Sec. I, the advantage of the Coulomb gauge is that the solution of Eq. (19a) is particularly simple to obtain, but its disadvantage is that the solution of Eq. (19b) is difficult to calculate.

Let us write the solutions of Eq. (19) in the explicit form

\[\Phi_C(x, t) = \frac{1}{4\pi\varepsilon_0} \int d^3\chi' \frac{1}{R} \rho(x', t), \quad (20a)\]

\[A_C(x, t) = \frac{\mu_0}{4\pi} \int d^3\chi' \frac{1}{R} \left[ J(x', t - R/c) - c \hat{R} \rho \times (x', t - R/c) \right] + \frac{c^2}{R} \int_0^{Rc} d\tau \rho(x', t - \tau). \quad (20b)\]

Equation (20a) is a well known instantaneous expression, and Eq. (20b) is a novel expression derived recently by Jackson.\(^\text{5}\) By making use of Eq. (20), Jackson\(^\text{5}\) obtained the retarded electric field in the form given by Jefimenko\(^\text{13}\) and the usual retarded form of the magnetic field in Eq. (13b). A disadvantage of the Coulomb gauge condition is that it cannot be written in a relativistically covariant form.

The scalar potential yields the instantaneous term

\[-\nabla \Phi_C(x, t) = \frac{1}{4\pi\varepsilon_0} \int d^3\chi' \frac{\hat{R}}{R^2} \rho(x', t). \quad (21)\]

This term does not display the properties of causality and propagation at the speed of light, and therefore its explicit presence in the expression for the retarded electric field \(E = -\nabla \Phi_C - \partial A_C / \partial t\) seems to indicate an inconsistency. We recall that an instantaneous field is in conflict with special relativity, which states that no physical information can propagate faster than \(c\) in vacuum.

To understand the role played by the acausal term \(-\nabla \Phi_C\) in the electric field \(E = -\nabla \Phi_C - \partial A_C / \partial t\), we apply the method discussed in Sec. III and show that the potentials \(\Phi_C\) and \(A_C\) lead to the fields \(E\) and \(B\).

In the first step we symmetrize Eq. (19a) with respect to Eq. (19b) by adding the term \(-(1/c^2)\partial^2 \Phi_C / \partial t^2\) on both sides of Eq. (19a) to obtain

\[\square^2 \Phi_C = -\frac{1}{\varepsilon_0} \rho - \frac{1}{c^2} \frac{\partial^2 \Phi_C}{\partial t^2}. \quad (22)\]

In the second step we take the negative of the gradient of Eq. (22) and the negative of the time derivative of Eq. (19b) to obtain two equations involving third-order derivatives of the potentials,

\[-\nabla^2 \Phi_C = \frac{1}{\varepsilon_0} \nabla \rho + \frac{1}{c^2} \frac{\partial^2 \Phi_C}{\partial t^2}, \quad (23a)\]

\[-\frac{\partial}{\partial t} \nabla \Phi_C = \frac{\mu_0}{\varepsilon_0} \frac{\partial J}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \Phi_C}{\partial t^2}. \quad (23b)\]

In the third step we add Eqs. (23a) and (23b) to obtain the wave equation

\[\square^2 \left( -\nabla \Phi_C - \frac{\partial A_C}{\partial t} \right) \frac{\partial A_C}{\partial t} \frac{\partial \Phi_C}{\partial t} \frac{\partial A_C}{\partial t} \frac{\partial \Phi_C}{\partial t}, \quad (24)\]

with the retarded solution

\[-\frac{\partial A_C}{\partial t} = -\frac{1}{4\pi\varepsilon_0} \int d^3\chi' \frac{1}{R} \left[ -\nabla' \rho - \frac{1}{c^2} \frac{\partial J}{\partial t'} \right] + \nabla \Phi_C. \quad (25)\]

Equation (25) shows that the term \(-\partial A_C / \partial t\) always contains the instantaneous component \(\Phi_C\), which cancels exactly.
the instantaneous part \( -\nabla \Phi_C \) of the electric field \( E = -\nabla \Phi_C - \partial A_C / \partial t \). This well known result has recently been emphasized.\(^ {5,14} \)

Equation (25) has also recently been obtained by applying another more complicated method.\(^ {15} \) The explicit presence of the acausal term \( -\nabla \Phi_C \) in the electric field is irrelevant because such a term is always canceled by one of the components \( \nabla \Phi_C \) of the remaining term defined by \( -\partial A_C / \partial t \). The field \( -\nabla \Phi_C \) is physically undetectable and can be interpreted as a spurious field. Hence, when Eq. (25) is used in \( E = -\nabla \Phi_C - \partial A_C / \partial t \), we obtain the usual retarded form of the electric field

\[
E = -\nabla \Phi_C - \frac{\partial A_C}{\partial t} = \frac{1}{4\pi\varepsilon_0} \int d^3x' \frac{1}{R} \left[ -\nabla' \rho - \frac{1}{c^2} \frac{\partial J}{\partial t'} \right].
\]

(26)

In the fourth step we take the curl of Eq. (19b) to obtain the wave equation

\[
\Box^2 (\nabla \times A_C) = -\mu_0 \nabla \times J.
\]

(27)

with the retarded solution

\[
\nabla \times A_C = \frac{\mu_0}{4\pi} \int d^3x' \frac{1}{R} (\nabla' \times J).
\]

(28)

Equation (28) is the usual retarded form of the magnetic field

\[
B = \nabla \times A_C = \frac{\mu_0}{4\pi} \int d^3x' \frac{1}{R} (\nabla' \times J).
\]

(29)

Note that neither the simple solution (20a) of Eq. (19a) nor the complicated solution (20b) of Eq. (19b) was required to show that the Coulomb gauge potentials yield the electric and magnetic fields.

V. THE FOUR STEPS OF THE METHOD

As noted in Sec. IV, the method can be defined by four steps:

1. Apply one of the Lorentz, Coulomb, Kirchhoff, velocity, and temporal gauges to Eq. (3) and symmetrize, if necessary, the gauged equation for the charge density with respect to the gauged equation for the current density.

2. Calculate the negative of the gradient of the gauged (and possibly symmetrized) equation for the charge density and the negative of the time derivative of the gauged equation for the current density. As a result, two equations containing third-order derivatives of potentials are obtained: one involving the gradient of the charge density and the other involving the time derivative of the current density.

3. Except for the Lorenz gauge, add the third-order equations obtained in step 2 to obtain a wave equation whose retarded solution gives an equation for the time derivative of the vector potential, which is substituted into the expression for the electric field in terms of the potentials to obtain the retarded electric field.

4. Take the curl of the third-order equation for the current density obtained in step 2 to obtain a wave equation whose retarded solution gives an equation for the curl of the vector potential, which is substituted into the expression for the magnetic field in terms of the vector potential to obtain the retarded magnetic field.

For the special case of the Lorentz gauge, it is necessary first to solve Eq. (14) derived in step 2. The retarded solutions of Eq. (15) are used in step 3 to obtain the usual retarded form of the electric field.\(^ {16} \)

VI. KIRCHHOFF GAUGE

According to Ref. 1, the first published relation between the potentials is due to Kirchhoff\(^ {10} \) who showed that the Weber form of the vector potential \( A \) and its associated scalar potential \( \Phi \) satisfies the equation (in modern notation),

\[
\nabla \cdot A = -1/(c^2) \partial \Phi / \partial t = 0, \tag{6}
\]

that is, Eq. (6), which was originally obtained for quasistatic potentials in which retardation is neglected. Of course, the electromagnetic gauge invariance had not been established at that time. The present author has proposed calling Eq. (6) the Kirchhoff gauge.\(^ {9} \) In the Kirchhoff gauge (6), Eq. (3) becomes

\[
\nabla^2 \Phi_K + \frac{1}{c^2} \frac{\partial^2 \Phi_K}{\partial t^2} = -\frac{1}{\varepsilon_0} \rho, \tag{30a}
\]

\[
\Box^2 A_K = -\mu_0 J + \frac{2}{c^2} \nabla \frac{\partial \Phi_K}{\partial t}. \tag{30b}
\]

We note that Eq. (30a) is an elliptical equation, which does not describe a real propagation. After the substitution \( c^2 = -(ic)^2 \), Eq. (30a) may be written as

\[
\nabla^2 \Phi_K - \frac{1}{(ic)^2} \frac{\partial^2 \Phi_K}{\partial t^2} = -\frac{1}{\varepsilon_0} \rho. \tag{31}
\]

Equation (31) formally states that \( \Phi_K \) propagates with the imaginary speed \( ic \), which emphasizes the unphysical character of \( \Phi_K \). The solutions of Eq. (30) can be expressed as\(^ {9} \)

\[
\Phi_K(x,t) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \frac{\rho(x',t - R/(ic))}{R}, \tag{32a}
\]

\[
A_K(x,t) = \frac{\mu_0}{4\pi} \int d^3x' \frac{1}{R} \left[ J(x',t - R/c) - c\hat{R} \rho \times (x',t - R/c) \right] + \frac{c^2\hat{R}}{i} \rho(x',t - R/(ic))
\]

\[
+ \frac{c^2\hat{R}}{R} \int_{R/(ic)}^{R} d\tau \rho(x',t - \tau). \tag{32b}
\]

It was noted in Ref. 9 that the potential \( \Phi_K \) in Eq. (32a) exhibits the same form as the scalar potential in the corresponding Lorenz gauge of an electromagnetic theory formulated in Euclidean four-space.\(^ {17,18} \) This interesting result shows how the same potential can be defined in different gauges and in different spacetimes. It is clear that the Kirchhoff gauge cannot be written in a relativistically covariant form. Also, in Ref. 9 it was shown that Eq. (32) yields the retarded electric and magnetic fields. However, as shown in Ref. 9, the derivation of Eq. (32) is not so simple.

We note that the potential \( \Phi_K \) yields the imaginary term\(^ {9} \)
\[-\nabla \Phi_K(x,t) = \frac{1}{4 \pi \varepsilon_0} \int d^3 x' \left( \frac{\hat{R}}{R^2} \delta(x', t - R/(ic)) + \hat{R} \frac{\partial \rho(x', t - R/(ic))}{\partial (t - R/(ic))} \right). \tag{33}\]

The reader might find surprising the presence of the instantaneous term \(-\nabla \Phi_K\) in the retarded expression for the electric field in terms of the Coulomb gauge potentials \(E = -\nabla \Phi_K - \partial A_K / \partial t\). The reader probably would find even more surprising the presence of the imaginary term \(-\nabla \Phi_K\) in the observable electric field in terms of the Kirchhoff gauge potentials \(E = -\nabla \Phi_K - \partial A_K / \partial t\). We suspect that, like the instantaneous term \(-\nabla \Phi_C\), the imaginary term \(-\nabla \Phi_K\) does not play a physical role in the electric field. To understand the role of \(-\nabla \Phi_K\) in Eq. (33), we will apply the four step method given in Sec. V. We will show that the Kirchhoff potentials \(\Phi_K\) and \(A_K\) lead to the retarded fields \(E\) and \(B\).

Step 1. After applying the Kirchhoff condition (6) to Eq. (3), we have already obtained Eq. (30). We then symmetrize Eq. (30a) with respect to Eq. (30b) by adding the term \(-2/c^2 \partial^2 \Phi_K / \partial t^2\) on both sides of Eq. (30a) to obtain

\[\Box^2 \Phi_K = - \frac{1}{\varepsilon_0} \rho - \frac{1}{c^2} \frac{\partial^2 \Phi_K}{\partial t^2}. \tag{34}\]

Step 2. We take the negative of the gradient of Eq. (34) and the negative of the time derivative of Eq. (30b) to obtain the equations

\[-\Box^2 \nabla \Phi_K = \frac{1}{\varepsilon_0} \nabla \rho + \frac{2}{c^2} \nabla \frac{\partial^2 \Phi_K}{\partial t^2}, \tag{35a}\]

\[-\Box^2 \frac{\partial A_K}{\partial t} = \mu_0 \frac{\partial J}{\partial t} + \frac{2}{c^2} \nabla \frac{\partial^2 \Phi_K}{\partial t^2}. \tag{35b}\]

Step 3. We add Eq. (35) to obtain the wave equation

\[\Box^2 \left( -\nabla \Phi_K - \frac{\partial A_K}{\partial t} \right) = \frac{1}{\varepsilon_0} \nabla \rho + \mu_0 \frac{\partial J}{\partial t}, \tag{36}\]

with the retarded solution

\[-\frac{\partial A_K}{\partial t} = - \frac{1}{4 \pi \varepsilon_0} \int d^3 x' \left[ -\nabla \rho - \frac{1}{c^2} \frac{\partial J}{\partial t} \right] + \nabla \Phi_K. \tag{37}\]

As can be seen, the term \(-\partial A_K / \partial t\) in Eq. (37) contains the imaginary component \(\nabla \Phi_K\), which exactly cancels the imaginary part \(-\nabla \Phi_K\) of the electric field \(E = -\nabla \Phi_K - \partial A_K / \partial t\). In other words, the explicit presence of the imaginary term \(-\nabla \Phi_K\) in the electric field is irrelevant because such a term is canceled by one of the components \(\nabla \Phi_K\) of the Kirchhoff-gauge vector potential \(-\partial A_K / \partial t\). This result has recently been emphasized.\(^{14}\) When Eq. (37) is used in the field \(E = -\nabla \Phi_K - \partial A_K / \partial t\), we obtain the usual retarded form of this field,

\[E = -\nabla \Phi_K - \frac{\partial A_K}{\partial t} = - \frac{1}{4 \pi \varepsilon_0} \int d^3 x' \left[ -\nabla \rho - \frac{1}{c^2} \frac{\partial J}{\partial t} \right]. \tag{38}\]

Step 4. We take the curl to Eq. (30b) to obtain the wave equation

\[\Box^2 (\nabla \times A_K) = - \mu_0 \nabla \times J, \tag{39}\]

with the retarded solution

\[\nabla \times A_K = \frac{\mu_0}{4 \pi} \int d^3 x' \frac{1}{R} [\nabla' \times J]. \tag{40}\]

Equation (40) is identified with the usual retarded form of the magnetic field

\[B = \nabla \times A_K = \frac{\mu_0}{4 \pi} \int d^3 x' \frac{1}{R} [\nabla' \times J]. \tag{41}\]

Therefore, we do not require the complicated Eq. (32) to verify that the Kirchhoff gauge potentials yield the retarded electric and magnetic fields.

**VII. VELOCITY GAUGE**

The velocity gauge (v-gauge) is one in which the scalar potential propagates with an arbitrary speed. This gauge is not well known despite the fact that it was proposed several years ago.\(^{12}\) The v-gauge is really a family of gauges that contains the Lorenz and Coulomb gauges as particular cases and also includes the Kirchhoff gauge.\(^{13,9}\) The v-gauge has recently been emphasized by Drury,\(^{19}\) Jackson,\(^{5}\) and Yang.\(^{11}\) If we assume the v-gauge defined by Eq. (7), then Eq. (3) become

\[\nabla^2 \Phi_v = - \frac{1}{v^2} \frac{\partial^2 \Phi_v}{\partial t^2} = - \frac{1}{\varepsilon_0} \rho, \tag{42a}\]

\[\Box^2 A_v = - \mu_0 J + \frac{1}{c^2} \nabla \left( 1 - \frac{c^2}{v^2} \right) \frac{\partial \Phi_v}{\partial t}. \tag{42b}\]

The generality of Eq. (42) becomes evident when we observe that it reduces to Eq. (9) when \(v = c\), to Eq. (19) when \(v = \infty\), and to Eq. (30) when \(v = ic\). The solutions of Eq. (42) are given by\(^5\)

\[\Phi_v(x,t) = \frac{1}{4 \pi \varepsilon_0} \int d^3 x' \frac{1}{R'} \rho(x', t - R/v), \tag{43a}\]

\[A_v(x,t) = \frac{\mu_0}{4 \pi} \int d^3 x' \frac{1}{R} \left( [J(x', t - R/c) - c \hat{R} \rho \times (x', t - R/c)] + \frac{c^2}{v} \hat{R} \rho(x', t - R/v) + \frac{c^2 \hat{R}}{R} \int_{R/v}^R d\rho \rho(x', t - \tau) \right). \tag{43b}\]

As expected, Eq. (43) reduces to Eq. (10) when \(v = c\), to Eq. (20) when \(v = \infty\), and to Eq. (32) when \(v = ic\). According to Eq. (42a), the potential \(\Phi_v\) propagates with an arbitrary speed \(v\), which may be subluminal \((v < c)\) or luminal \((v = c)\) or superluminal \((v > c)\) including the instantaneous limit \((v = \infty)\). In Ref. 5 it was shown that Eq. (43) yields the retarded electric and magnetic fields. Jackson has emphasized.\(^5\) “The v-gauge illustrates dramatically how arbitrary and unphysical the potentials can be, yet still yield the same physically sensible fields.” The velocity gauge cannot be written in a relativistically covariant form.

The v-gauge scalar potential \(\Phi_v\) generates the field\(^5\)
\[ -\nabla \Phi_v(x,t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \left( \frac{\hat{R}}{R} \rho(x',t-R/v) + \frac{\hat{R}}{Rv} \frac{\partial \rho(x',t-R/v)}{\partial t} \right). \] (44)

This term does not display the property of propagation at the speed of light \( c \). In particular, when \( v \) is superluminal, we have a conflict with special relativity. We suspect that the term \(-\nabla \Phi_v\) does not play a physical role in the electric field.

To understand the role played by the term \(-\nabla \Phi_v\) in the field \( \mathbf{E} = -\nabla \Phi_v - \partial \mathbf{A}_v / \partial t \), we apply the four step method to show that the potentials \( \Phi_v \) and \( \mathbf{A}_v \) yield the fields \( \mathbf{E} \) and \( \mathbf{B} \).

Step 1. If we apply the velocity condition (7) to Eq. (3), we obtain Eq. (42). We symmetrize Eq. (42a) with respect to Eq. (42b) by adding the term \(-11/c^2\)\( \partial^2 \Phi_v / \partial t^2 \) on both sides of Eq. (42a). The resulting equation can be written as

\[ \square^2 \Phi_v = -\frac{1}{\epsilon_0} \rho - \frac{1}{c^2} \left(1 - \frac{v^2}{c^2} \right) \partial^2 \Phi_v. \] (45)

Step 2. We take minus the gradient of Eq. (45) and minus the time derivative of Eq. (42b). The resulting equations are

\[ -\square^2 \nabla \Phi_v = \frac{1}{\epsilon_0} \nabla \rho + \frac{1}{c^2} \nabla \left(1 - \frac{v^2}{c^2} \right) \partial^2 \Phi_v, \] (46a)

\[ -\square^2 \frac{\partial \mathbf{A}_v}{\partial t} = \frac{\mu_0}{\epsilon_0} \frac{\partial J}{\partial t} + \frac{1}{c^2} \nabla \left(1 - \frac{v^2}{c^2} \right) \partial^2 \Phi_v. \] (46b)

Step 3. We add Eq. (46) to obtain the wave equation

\[ \square^2 \left(-\nabla \Phi_v - \frac{\partial \mathbf{A}_v}{\partial t} \right) = \frac{1}{\epsilon_0} \nabla \rho + \mu_0 \frac{\partial J}{\partial t}, \] (47)

with the retarded solution

\[ -\frac{\partial \mathbf{A}_v}{\partial t} = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{1}{R} \left[ -\nabla' \rho - \frac{1}{c^2} \frac{\partial J}{\partial t} \right] + \nabla \Phi_v. \] (48)

We observe that the term \(-\partial \mathbf{A}_v / \partial t \) in Eq. (48) contains the component \( \nabla \Phi_v \), which cancels the term \(-\nabla \Phi_v \). In other words, the explicit presence of a term possessing an arbitrary propagation \(-\nabla \Phi_v \) in the electric field is irrelevant because such a term is canceled by one of the components \( \nabla \Phi_v \) of the \( v \)-gauge vector potential \( (-\partial \mathbf{A}_v / \partial t \) ). This cancellation means that the field \( -\nabla \Phi_v \) is spurious and the propagation at the speed of light \( c \) of the electric field is not lost. Therefore, when Eq. (48) is used in the expression \( \mathbf{E} = -\nabla \Phi_v - \partial \mathbf{A}_v / \partial t \), we obtain the usual retarded form of the electric field

\[ \mathbf{E} = -\nabla \Phi_v - \frac{\partial \mathbf{A}_v}{\partial t} = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{1}{R} \left[ -\nabla' \rho - \frac{1}{c^2} \frac{\partial J}{\partial t} \right]. \] (49)

Step 4. We take the curl of Eq. (42b) to obtain the wave equation

\[ \square^2 (\mathbf{A}_v) = -\mu_0 \nabla \times \mathbf{J}, \] (50)

with the retarded solution

\[ \nabla \times \mathbf{A}_v = \frac{\mu_0}{4\pi\epsilon_0} \int d^3x' \frac{1}{R} (\nabla' \times \mathbf{J}). \] (51)

Equation (51) gives the usual retarded form of the magnetic field

\[ \mathbf{B} = \nabla \times \mathbf{A}_v = \frac{\mu_0}{4\pi\epsilon_0} \int d^3x' \frac{1}{R} (\nabla' \times \mathbf{J}). \] (52)

We see that we do not require the complicated Eq. (43) to show that the potentials in the velocity gauge yield the retarded electric and magnetic fields.

**VIII. TEMPORAL GAUGE**

We have pointed out that the scalar potential can be instantaneous, imaginary, and superluminal depending on the gauge (Coulomb, Kirchhoff, and velocity gauges, respectively). Now we will see that the scalar potential can also not exist. The temporal gauge is one in which the scalar potential is identically zero, which means that the electric and magnetic fields are defined only by the vector potential:

\[ \mathbf{E} = -\frac{\partial \mathbf{A}_T}{\partial t}, \] (53a)

\[ \mathbf{B} = \nabla \times \mathbf{A}_T. \] (53b)

The temporal gauge cannot be written in a relativistically covariant form. The reader might wonder why the scalar potential in the temporal gauge does not exist, despite the fact that there is a nonzero charge density. The simple answer is that the values of the charge density do not necessarily lead to a scalar potential in all gauges. The existence of a scalar potential generally depends on the adopted gauge. In other words, the retarded values of the charge density always contribute physically to the electric field, but they do not lead to a scalar potential in the temporal gauge.

If we assume the temporal gauge defined by Eq. (8), then Eq. (3) becomes

\[ \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}_T) = -\frac{1}{\epsilon_0} \rho, \] (54a)

\[ \square^2 \mathbf{A}_T = -\mu_0 \mathbf{J} + \nabla (\nabla \cdot \mathbf{A}_T). \] (54b)

The solution of Eq. (54) is given by

\[ \mathbf{A}_T(x,t) = \frac{\mu_0}{4\pi} \int d^3x' \frac{1}{R} \left[ (\mathbf{J}(x',t-R/c) - c \hat{R} \rho \times (x',t-R/c) - \frac{c^2 R}{R} \int_{R/c}^{R/c} d\tau \rho(x',t-\tau) \right]. \] (55)

In Ref. 5 it was demonstrated that the potential \( \mathbf{A}_T \) in Eq. (55) yields the retarded electric and magnetic fields. To understand this result, we apply our method to show that \( \mathbf{A}_T \) yields the retarded fields \( \mathbf{E} \) and \( \mathbf{B} \).

Step 1. After applying the temporal condition (8) to Eq. (3), we obtain Eq. (54). It is not necessary to symmetrize Eq. (54a) with respect to Eq. (54b).

Step 2. We take minus the gradient of Eq. (54a) and minus the time derivative of Eq. (54b) to obtain
\[- \frac{\partial}{\partial t} \nabla (\nabla \cdot A_T) = \frac{1}{\epsilon_0} \nabla \rho, \quad (56a)\]

\[- \nabla^2 \frac{\partial A_T}{\partial t} = \mu_0 \frac{\partial J}{\partial t} - \nabla \left( \nabla \cdot A_T \right). \quad (56b)\]

Step 3. We add Eq. (56) to obtain the wave equation
\[\nabla^2 \left( \frac{\partial A_T}{\partial t} \right) = \frac{1}{\epsilon_0} \nabla \rho + \mu_0 \frac{\partial J}{\partial t}, \quad (57)\]

with the retarded solution
\[- \frac{\partial A_T}{\partial t} = \frac{1}{4\pi\epsilon_0} \int d^3x' \left[ \frac{1}{R} \left( -\nabla' \rho + \frac{1}{c^2} \frac{\partial J}{\partial t'} \right) \right]. \quad (58)\]

In the temporal gauge there is no unphysical term on the right-hand side of Eq. (58). When Eq. (58) is used in the expression \(E = -\frac{\partial A_T}{\partial t}\), we obtain the usual retarded form of the electric field
\[E = -\frac{\partial A_T}{\partial t} = \frac{1}{4\pi\epsilon_0} \int d^3x' \left[ \frac{1}{R} \left( -\nabla' \rho - \frac{1}{c^2} \frac{\partial J}{\partial t'} \right) \right]. \quad (59)\]

Step 4. We take the curl of Eq. (54b) to obtain the wave equation
\[\nabla \times \nabla \times A_T = -\mu_0 \nabla \times J, \quad (60)\]

with the retarded solution
\[\nabla \times A_T = \frac{\mu_0}{4\pi} \int d^3x' \left[ \frac{1}{R} \left( \nabla' \times J \right) \right], \quad (61)\]

which directly leads to the usual retarded form of the magnetic field
\[B = \nabla \times A_T = \frac{\mu_0}{4\pi} \int d^3x' \left[ \frac{1}{R} \left( \nabla' \times J \right) \right]. \quad (62)\]

The complicated form of Eq. (55) is not required to show that the vector potential in the temporal gauge yields the retarded electric and magnetic fields.

**IX. THE NONSPURIOUS CHARACTER OF \(-\nabla \Phi_L\)**

The fact that the Coulomb gauge scalar potential \(\Phi_C\) propagates instantaneously is of no concern if we assume that the electromagnetic potentials are not physically measurable quantities. As pointed out by Griffiths,\(^2\) “The point is that \(V\) [the Coulomb-gauge scalar potential] by itself is not a physically measurable quantity.” It follows that the instantaneous term \(-\nabla \Phi_C\) [Eq. (21)] must also be an unphysical quantity. The subtle point is that \(-\nabla \Phi_C\) is part of the physical electric field expressed in terms of the Coulomb gauge potentials: \(E = -\nabla \Phi_C - \partial A_C/\partial t\). We have pointed out that the presence of \(-\nabla \Phi_C\) in the electric field is irrelevant because it is canceled by a component of the remaining term \(-\partial A_C/\partial t\). We have noted that \(-\nabla \Phi_C\) is a formal result of the theory and has no physical meaning. We have drawn similar conclusions for the term with imaginary propagation \(-\nabla \Phi_K\) [Eq. (33)] and for the term with arbitrary propagation \(-\nabla \Phi_L\) [Eq. (44)]. We cannot come to the same conclusion for the term \(-\nabla \Phi_L\) [Eq. (11)] due to the Lorenz gauge potential \(\Phi_L\), because this term displays the experimentally verified properties of causality and propagation at the speed of light—the term \(-\nabla \Phi_L\) is not canceled by part of the remaining term \(-\partial A_L/\partial t\) of the electric field.

Therefore, we can conclude that \(-\nabla \Phi_L\) is not a spurious term like the terms \(-\nabla \Phi_C\), \(-\nabla \Phi_K\), and \(-\nabla \Phi_K\). It follows that \(-\nabla \Phi_L\) can be interpreted as a physical quantity. Similarly, the term \(-\partial A_L/\partial t\) also satisfies the properties of causality and propagation at the speed of light, which indicates that this term should not be interpreted as being spurious. Thus, \(-\partial A_L/\partial t\) can also be interpreted as a physical quantity. The physical character of each one of the terms \(-\nabla \Phi_L\) and \(-\partial A_L/\partial t\) is strongly supported by the fact that the combination \(-\nabla \Phi_L - \partial A_L/\partial t\), that is, the electric field, is physically detectable.

The Lorenz gauge potentials \(\Phi_L\) and \(A_L\) naturally yield the electric and magnetic fields with the physical properties of causality and propagation at the speed of light, which suggests that the Lorenz gauge potentials (and not the Coulomb, Kirchhoff, and velocity potentials) can be interpreted as physical quantities.

**X. CONCLUDING REMARKS**

Jackson has pointed out that:\(^5\) “It seems necessary from time to time to show that the electric and magnetic fields are independent of the choice of gauge for the potentials.” We have demonstrated that the fields are independent of the choice of gauge for potentials in a variety of gauges (Lorentz, Coulomb, Kirchhoff, velocity, and temporal gauges). Our proposed method can be used to demonstrate that the potentials in these gauge yield the same retarded electric and magnetic fields. Instead of using explicit expressions for the scalar and vector potentials in the various gauges, the method uses the dynamical equations of these potentials to obtain two wave equations, whose retarded solutions lead to the retarded fields. We identified the spurious character of the gradient of the scalar potential in the Coulomb, Kirchhoff, and velocity gauges and have emphasized the nonspurious character of the scalar potential in the Lorenz gauge. Finally, we suggested that the Lorenz gauge potentials can be interpreted as physical quantities.

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\(^{a}\)Electronic mail: heras@phys.lsu.edu


16 In the proposed method we could also choose the advanced solutions (those evaluated at the advanced time \( t' = t + R/c \)) of the wave equations derived in steps 3 and 4. This choice would lead to the advanced form of the electric and magnetic fields.
20 Even though the combination \(-\nabla \Phi_2 - \partial A_v / \partial t\) can be detected, it can be argued that this possibility does not necessarily imply that each term can be detected separately.