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Alternative routes to the retarded potentials

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Abstract

Two procedures to introduce the familiar retarded potentials of Maxwell's equations are reviewed. The first well-known procedure makes use of the Lorenz-gauge potentials of Maxwell's equations. The second less-known procedure applies the retarded Helmholtz theorem to Maxwell's equations. Both procedures are compared in the context of an undergraduate presentation of electrodynamics. The covariant form of both procedures is discussed for completeness. As a related discussion, two procedures to introduce the unfamiliar instantaneous potentials of Maxwell's equations are also reviewed. The first procedure applies the standard Helmholtz theorem to Maxwell's equations and the second one uses the Coulomb-gauge potentials of Maxwell's equations. The retarded and instantaneous forms of the potentials of Maxwell's equations are briefly commented upon. The retarded Helmholtz theorem is used to introduce the retarded potentials of Maxwell's equations with magnetic monopoles.

Keywords: Maxwell's equations, Helmholtz theorem, gauge invariance

1. Introduction

There are several equivalent procedures to introduce the familiar retarded potentials of Maxwell's equations [1–4]. One of these procedures, and perhaps the most popular, is the traditional one [1] in which one fixes the Lorenz-gauge condition

$$\nabla \cdot \mathbf{A}_L + \frac{1}{c^2}\frac{\partial \Phi_L}{\partial t} = 0, \tag{1}$$

(SI units will be used in this paper) in the coupled equations for the scalar and vector potentials, which have been derived in turn from Maxwell's equations. As a consequence, the equations for the potentials are decoupled and thus one finally has two wave equations: $\Box^2\Phi_L = -\rho/\epsilon_0$ and $\Box^2\mathbf{A}_L = -\mu_0\mathbf{J}$, where $\Box^2 \equiv \nabla^2 - (1/c^2)\partial^2/\partial t^2$ is the d'Alembert operator, and $\rho = \rho(\mathbf{r}, t)$ and $\mathbf{J} = \mathbf{J}(\mathbf{r}, t)$ are the charge and current densities. These wave equations are then solved by assuming appropriate boundary conditions and the obtained solutions Φ_L and A_L are identified with the retarded scalar and vector potentials. Using these potentials, one corroborates (1) *a posteriori*.

There is another procedure to obtain the retarded potentials in which one first applies a retarded form of the Helmholtz theorem [5–9] to the fields of Maxwell's equations. The theorem requires the specification of appropriate boundary conditions for the electric and magnetic fields. Via this procedure, one first obtains retarded expressions for electric and magnetic fields from which one 'extracts' the retarded scalar and vector potentials 'by inspection'. One then shows *a posteriori* that these retarded potentials satisfy the equation

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0 \tag{2}$$

and also the wave equations $\Box^2\Phi = -\rho/\epsilon_0$ and $\Box^2\mathbf{A} = -\mu_0\mathbf{J}$, which are the same as those appearing in the traditional procedure. This indirect procedure to obtain the retarded potentials from the retarded fields has been little explored in the literature [10, 11]. Hopefully, a comparison between both procedures may be pedagogically useful in an undergraduate presentation of electrodynamics.

There are also two equivalent methods to introduce instantaneous forms of scalar and vector potentials of Maxwell's equations. These instantaneous potentials are rarely mentioned in textbooks (see, for example, pp 84–8 in [3]), despite their historical importance. In the first method one fixes the Coulomb-gauge condition

$$\nabla \cdot \mathbf{A}_C = 0, \tag{3}$$

in the coupled equations for potentials derived from Maxwell's equations, obtaining the instantaneous Poisson equations $\nabla^2\Phi_C = -\rho/\epsilon_0$ and $\nabla^2\mathbf{A}_C = -\mu_0\mathbf{J} - (1/c^2)\partial\mathbf{E}/\partial t$. By assuming appropriate boundary conditions, one finds integral (instantaneous) representations for the scalar and vector potentials. Using these potentials one verifies (3) *a posteriori*.

In the second method one applies the standard, as opposed to the retarded Helmholtz theorem, to Maxwell's equations, obtaining instantaneous expressions for the electric and magnetic fields, from which one extracts instantaneous scalar and vector potentials by inspection. *A posteriori* one shows that the instantaneous vector potential satisfies

$$\nabla \cdot \mathbf{A} = 0 \tag{4}$$

and that the instantaneous potentials satisfy the Poisson equations $\nabla^2\Phi = -\rho/\epsilon_0$ and $\nabla^2\mathbf{A} = -\mu_0\mathbf{J} - (1/c^2)\partial\mathbf{E}/\partial t$, which are the same as those appearing in the first procedure.

This paper discusses in some detail the above outlined procedures to introduce retarded potentials by emphasising the main pedagogical advantages of each procedure in an undergraduate presentation of electrodynamics. As a complementary material, the two outlined methods to introduce the little-known instantaneous potentials are also discussed in some detail. The paper is organised as follows. The traditional procedure based on the use of the gauge invariance through the adoption of the Lorenz-gauge condition is reviewed in section 2. The novel procedure based on the retarded Helmholtz theorem is discussed in section 3. The traditional and novel procedures are formulated in the four-dimensional Minkowski space-time in section 4. Advantages of both procedures are enlighten in section 5. For completeness, the analogous procedures to derive instantaneous forms of the scalar and vector potentials are discussed in section 6. The instantaneous and retarded versions of the potentials are briefly commented in section 7 in which some historical remarks are made. An appendix is given in which the procedure based on the retarded Helmholtz theorem is applied to introduce the retarded potentials of Maxwell's equations with magnetic monopoles.

2. Gauge invariance and retarded potentials

Suppose one knows Maxwell's equations and their gauge invariance but does not know the retarded Helmholtz theorem. The homogeneous (sourceless) Maxwell's equations imply the existence of the potentials Φ and $\mathbf{A}$ such that $\mathbf{E} = -\nabla\Phi - \partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla \times \mathbf{A}$. Using these expressions in the inhomogeneous (source) Maxwell's equations, one obtains the coupled equations for the potentials

$$\nabla^2\Phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0}, \tag{5}$$

$$\Box^2\mathbf{A} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right) = -\mu_0\mathbf{J}. \tag{6}$$

By adding the identically zero term: $(1/c^2)\partial^2\Phi/\partial t^2 - (1/c^2)\partial\Phi/\partial t^2 \equiv 0$ to the left of (5) it follows that the coupled equations are symmetrized [12]:

$$\Box^2\Phi + \frac{\partial}{\partial t}\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right) = -\frac{\rho}{\epsilon_0}, \tag{7}$$

$$\Box^2\mathbf{A} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right) = -\mu_0\mathbf{J}. \tag{8}$$

These equations can be uncoupled by exploiting the invariance of the fields $\mathbf{E}$ and $\mathbf{B}$ under the gauge transformations

$$\Phi \longrightarrow \Phi' = \Phi - \frac{\partial\Lambda}{\partial t}, \quad \mathbf{A} \longrightarrow \mathbf{A}' = \mathbf{A} + \nabla\Lambda, \tag{9}$$

where $\Lambda = \Lambda(\mathbf{r}, t)$ is an arbitrary gauge function. In fact, the arbitrariness of Λ allows one to chose a set of potentials satisfying the condition

$$\nabla \cdot \mathbf{A}_L + \frac{1}{c^2}\frac{\partial\Phi_L}{\partial t} = 0, \tag{10}$$

which uncouples equations (7) and (8)and then one obtains two wave equations

$$\Box^2\Phi_L = -\frac{\rho}{\epsilon_0}, \quad \Box^2\mathbf{A}_L = -\mu_0\mathbf{J}, \tag{11}$$

whose retarded solutions give the retarded potentials

$$\Phi_L = \frac{1}{4\pi\epsilon_0}\int\frac{[\rho]}{R}\mathrm{d}^3r', \quad \mathbf{A}_L = \frac{\mu_0}{4\pi}\int\frac{[\mathbf{J}]}{R}\mathrm{d}^3r', \tag{12}$$

where the integrals are extended over all space, $R = |\mathbf{r} - \mathbf{r}'|$ with $\mathbf{r}$ being the field point and $\mathbf{r}'$ the source point, and the square bracket [ ] means that the enclosed quantity is to be evaluated at the retarded time $t' = t - R/c$. The relation (10) is called the Lorenz-gauge condition and Φ_L and $\mathbf{A}_L$ are the Lorenz-gauge potentials. To show that potentials Φ and $\mathbf{A}$

can always be found to satisfy (10), suppose that our original potentials Φ and $\mathbf{A}$ satisfy (7) and (8) but do not satisfy (10), that is, $\nabla\cdot\mathbf{A}+(1/c^2)\partial\Phi/\partial t=g\neq 0$, where g is a known scalar function of space and time. Since Λ is an arbitrary function one can always demand it satisfies the wave[1] equation: $\Box^2\Lambda=-g$, which guarantees the validity of the Lorenz condition for the transformed potentials: $\nabla\cdot\mathbf{A}'+(1/c^2)\partial\Phi'/\partial t=0$. In fact, using the transformations (9) in the expression $\nabla\cdot\mathbf{A}'+(1/c^2)\partial\Phi'/\partial t$ it follows that

$$\nabla\cdot\mathbf{A}'+\frac{1}{c^2}\frac{\partial\Phi'}{\partial t}=\nabla\cdot(\mathbf{A}+\nabla\Lambda)+\frac{1}{c^2}\frac{\partial}{\partial t}\left(\Phi-\frac{\partial\Lambda}{\partial t}\right)=g+\Box^2\Lambda=0. \tag{13}$$

As a final point one can directly verify that the potentials Φ_L and $\mathbf{A}_L$ given in (12) satisfy (10) (see (19) for the explicit calculation).

3. Retarded Helmholtz theorem and retarded potentials

Suppose one knows the retarded Helmholtz theorem [5–9] and Maxwell's equations but does not know the gauge invariance of these equations. One defines a retarded (causal) vector field as a time-dependent vector field $\mathbf{F}(\mathbf{r},t)$ whose sources are evaluated at the retarded time $t'=t-R/c$ in agreement with the causality principle[2]. The retarded Helmholtz theorem states that any retarded vector field $\mathbf{F}(\mathbf{r},t)$ that goes to zero faster than $1/r$ as $r\to\infty$ can be expressed as [5–9]:

$$\mathbf{F}=-\nabla\int\frac{[\nabla'\cdot\mathbf{F}]}{4\pi R}\,\mathrm{d}^3r'+\nabla\times\int\frac{[\nabla'\times\mathbf{F}]}{4\pi R}\,\mathrm{d}^3r'+\frac{1}{c^2}\frac{\partial}{\partial t}\int\frac{[\partial\mathbf{F}/\partial t]}{4\pi R}\,\mathrm{d}^3r'. \tag{14}$$

As may be seen, the formal sources of the field $\mathbf{F}$ are its divergence, curl and time derivative, all of them evaluated at the retarded time. If (14) is applied to Maxwell's equations and an appropriate integration by parts made[3], then one obtains the retarded fields [5–9]:

$$\mathbf{E}=-\nabla\left\{\frac{1}{4\pi\epsilon_0}\int\frac{[\rho]}{R}\mathrm{d}^3r'\right\}-\frac{\partial}{\partial t}\left\{\frac{\mu_0}{4\pi}\int\frac{[\mathbf{J}]}{R}\mathrm{d}^3r'\right\}, \tag{15}$$

$$\mathbf{B}=\nabla\times\left\{\frac{\mu_0}{4\pi}\int\frac{[\mathbf{J}]}{R}\mathrm{d}^3r'\right\}. \tag{16}$$

From these expressions one can 'extract' the retarded scalar and vector potentials:

$$\Phi=\frac{1}{4\pi\epsilon_0}\int\frac{[\rho]}{R}\mathrm{d}^3r',\qquad \mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{[\mathbf{J}]}{R}\mathrm{d}^3r', \tag{17}$$

in terms of which (15) and (16) take the compact form $\mathbf{E}=-\nabla\Phi-\partial\mathbf{A}/\partial t$ and $\mathbf{B}=\nabla\times\mathbf{A}$. The retarded potentials in (17) are seen to satisfy the equation

$$\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}=0. \tag{18}$$

1 A retarded solution of this equation reads $\Lambda=\int\{[g]/(4\pi R)\}\,\mathrm{d}^3r'$.

2 The notation $\mathbf{F}(\mathbf{r},t)$ is not unambiguous to express a retarded field. One knows that $\mathbf{F}(\mathbf{r},t)$ is a retarded field only when one observes that their sources are evaluated at the retarded time $t'=t-R/c$.

3 In this integration one uses the result $\nabla\times\int\{[\partial\mathbf{F}/\partial t]/R\}\,\mathrm{d}^3r'=(\partial/\partial t)\int\{[\nabla'\times\mathbf{F}]/R\}\,\mathrm{d}^3r'$.

In fact, from the potentials in (17) and the continuity equation it follows that

$$\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}=\frac{\mu_0}{4\pi}\int\nabla\cdot\left(\frac{[\mathbf{J}]}{R}\right)\mathrm{d}^3r'+\frac{1}{4\pi\epsilon_0c^2}\int\frac{\partial}{\partial t}\left(\frac{[\rho]}{R}\right)\mathrm{d}^3r'$$
$$=\frac{\mu_0}{4\pi}\int\frac{1}{R}\left[\nabla'\cdot\mathbf{J}+\frac{\partial\rho}{\partial t'}\right]\mathrm{d}^3r'=0, \qquad (19)$$

where $[\nabla'\cdot\mathbf{J}]/R=\nabla\cdot([\mathbf{J}]/R)+\nabla'\cdot([\mathbf{J}]/R)$, $[\partial\rho/\partial t']=\partial[\rho]/\partial t$ and the fact that the surface integral originated by the term $\nabla'\cdot([\mathbf{J}]/R)$ vanishes at space infinity have been used. Similarly, if one takes the d'Alembert operator $\Box^2$ to the potentials in (17) and uses [13]: $\Box^2([\boldsymbol{\mathcal{X}}]/R)=-4\pi[\boldsymbol{\mathcal{X}}]\delta(\mathbf{r}-\mathbf{r}')$[4], then one arrives at the wave equations

$$\Box^2\Phi=\frac{1}{4\pi\epsilon_0}\int\Box^2\left(\frac{[\rho]}{R}\right)\mathrm{d}^3r'=-\frac{\rho}{\epsilon_0}, \qquad (20)$$

$$\Box^2\mathbf{A}=\frac{\mu_0}{4\pi}\int\Box^2\left(\frac{[\mathbf{J}]}{R}\right)\mathrm{d}^3r'=-\mu_0\mathbf{J}. \qquad (21)$$

Equations (12) and (17) describe the same retarded potentials ($\Phi_L=\Phi$ and $\mathbf{A}_L=\mathbf{A}$) which satisfy the same equations and therefore the equivalence between the procedures discussed in sections 2 and 3 is established.

In the process of extracting the retarded potentials Φ and $\mathbf{A}$ from the retarded fields $\mathbf{E}$ and $\mathbf{B}$ by inspection, one has implicitly imposed the Lorenz-gauge condition. Accordingly, (18) should not be interpreted as an additional field equation but as a gauge fixing condition. Although the definitions of Φ and $\mathbf{A}$ in (17) may seem natural, one is not forced to make such definitions. For example, one could add $\nabla\Lambda$, where $\Lambda(\mathbf{r}, t)$ is an arbitrary scalar function, to the expression between the parentheses { } of (16) and the magnetic field remains unchanged. When one extracts the retarded potentials from the retarded fields by inspection one tacitly assumes $\Lambda=0$, i.e. one implicitly adopts the Lorenz-gauge condition.

4. The two procedures in the Minkowski space-time

Suppose one knows the covariant form of Maxwell's equations expressed in terms of the four-potential and its associated gauge invariance but does not know the covariant form of the retarded Helmholtz theorem for antisymmetric tensors [9, 14, 15]. Greek indices μ, ν, $\kappa\ldots$ run from 0 to 3. The summation convention on repeated indices is adopted. The signature of the metric is $(+, -, -, -)$. A point is denoted by $x=x^\mu$. The totally antisymmetric four-dimensional tensor reads $\varepsilon^{\mu\nu\alpha\beta}$ with $\varepsilon^{0123}=1$.

The homogeneous Maxwell equation $\partial_\mu\,{}^*\!F^{\mu\nu}=0$, where ${}^*\!F^{\mu\nu}=(1/2)\varepsilon^{\mu\nu\kappa\lambda}F_{\kappa\lambda}$ is the dual of the electromagnetic field $F^{\mu\nu}$, implies the existence of the four-potential A^ν such that $F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu$, which is inserted in the inhomogeneous Maxwell equation $\partial_\mu F^{\mu\nu}=\mu_0J^\nu$, where J^ν is the four-current, to obtain the equation

$$\partial_\mu\partial^\mu A^\nu-\partial^\nu\partial_\mu A^\mu=\mu_0J^\nu. \qquad (22)$$

[4] This nice identity is true for functions $\boldsymbol{\mathcal{X}}$ such that the quantities $[\boldsymbol{\mathcal{X}}]/R$ have not the form $[\boldsymbol{\mathcal{X}}]/R=f(R)[\mathbf{F}]$. If for example $[\boldsymbol{\mathcal{X}}]/R=R[\mathbf{F}]$ then $\Box^2(R[\mathbf{F}])=0$ since $-4\pi R[\mathbf{F}]\delta(\mathbf{r}-\mathbf{r}')$ vanishes for $\mathbf{r}\neq\mathbf{r}'$ because of the delta function and also for $\mathbf{r}=\mathbf{r}'$ because this equality implies $R=0$.

This equation is invariant under the gauge transformation

$$A^{\mu} \longrightarrow A'^{\mu} = A^{\mu} + \partial^{\mu}\Lambda, \tag{23}$$

where $\Lambda = \Lambda(x)$ is an arbitrary gauge function of space-time. The arbitrariness of Λ allows us to chose a four-potential such that

$$\partial_{\mu}A_L^{\mu} = 0. \tag{24}$$

If this condition is used in (22) it reduces to the wave equation

$$\partial_{\mu}\partial^{\mu}A_L^{\nu} = \mu_0 J^{\nu}, \tag{25}$$

whose retarded solution is

$$A_L^{\nu} = \mu_0 \int G J^{\nu}(x')\mathrm{d}^4x'. \tag{26}$$

where $G = \{ct - ct' - R\}(4\pi R) = \{t - t' - R/c\}/(4\pi Rc)$. is the retarded Green function of the wave equation, d^4x' is a volume element in space-time and the integral is over all space-time. Equation (26) represents the covariant form of the retarded potentials.

Analogously, the relation (24) represents the covariant form of the Lorenz-gauge condition. To show that a potential A^{μ} can always be found to satisfy (24), suppose that A^{μ} satisfies (22) but does not satisfy (24), that is, $\partial_{\mu}A^{\mu} = g \neq 0$, where g is a scalar function. Since Λ is an arbitrary function one can demand the equation $\partial_{\mu}\partial^{\mu}\Lambda = -g$, which guarantees the validity of the Lorenz condition for the transformed four-potential: $\partial_{\mu}A'^{\mu} = 0$. The proof is as follows. Using the gauge transformation (23) in $\partial_{\mu}A'^{\mu}$ one obtains

$$\partial_{\mu}A'^{\mu} = \partial_{\mu}(A^{\mu} + \partial^{\mu}\Lambda) = g + \partial_{\mu}\partial^{\mu}\Lambda = 0. \tag{27}$$

Suppose now one knows the covariant form of the retarded Helmholtz theorem for antisymmetric tensors [9, 14, 15] and the covariant form of Maxwell's equations but does not know the gauge invariance of these equations. The retarded Helmholtz theorem for antisymmetric tensor fields in the Minkowski space-time states that a retarded antisymmetric tensor field $\mathcal{F}^{\mu\nu}(x)$ vanishing sufficiently rapidly at spatial infinity is completely determined by specifying its divergence $\partial_{\alpha}\mathcal{F}^{\alpha\lambda}$ and the divergence of its dual $\partial_{\alpha}{}^{*}\mathcal{F}^{\alpha\lambda}$. According to this theorem the field tensor can then be expressed as:

$$\mathcal{F}^{\mu\nu}(x) = \partial^{\mu\nu}{}_{\lambda}\int G\partial'_{\alpha}\mathcal{F}^{\alpha\lambda}(x')\mathrm{d}^4x' - {}^{*}\partial^{\mu\nu}{}_{\lambda}\int G\partial'_{\alpha}{}^{*}\mathcal{F}^{\alpha\lambda}(x')\mathrm{d}^4x', \tag{28}$$

where $\partial^{\mu\nu}{}_{\lambda}$ is an operator antisymmetric in μ and ν defined as $\partial^{\mu\nu}{}_{\lambda} = \delta^{\nu}_{\lambda}\partial^{\mu} - \delta^{\mu}_{\lambda}\partial^{\nu}$ and its associated dual is given by ${}^{*}\partial^{\mu\nu}{}_{\lambda} = (1/2)\varepsilon^{\mu\nu\alpha\beta}\partial_{\alpha\beta\lambda}$[5]. The theorem in (28) can directly be applied to the covariant form of Maxwell's equations: $\partial_{\mu}F^{\mu\nu} = \mu_0 J^{\nu}$ and $\partial_{\mu}{}^{*}F^{\mu\nu} = 0$. In fact, if one makes the identification $\mathcal{F}^{\mu\nu} = F^{\mu\nu}$ and use Maxwell's equations then one obtains the retarded electromagnetic field

$$F^{\mu\nu} = \mu_0\, \partial^{\mu\nu}{}_{\lambda}\int G J^{\lambda}(x')\mathrm{d}^4x'. \tag{29}$$

If the four-potential A^{λ} is defined as

$$A^{\lambda} = \mu_0 \int G J^{\lambda}(x')\mathrm{d}^4x', \tag{30}$$

then the field $F^{\mu\nu}$ takes its familiar form $F^{\mu\nu} = \partial^{\mu\nu}{}_{\lambda}A^{\lambda} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$. The four-potential A^{λ} in (30) satisfies the equation

[5] These operators satisfy ${}^{*}\partial^{\mu\nu}{}_{\lambda} = \varepsilon^{\mu\nu\alpha}{}_{\lambda}\partial_{\alpha}$, $\partial_{\mu}\partial^{\mu\nu}{}_{\lambda} = \delta^{\nu}_{\lambda}\partial_{\mu}\partial^{\mu} - \partial_{\lambda}\partial^{\nu}$,$^{**}\partial^{\mu\nu}{}_{\lambda} = -\partial^{\mu\nu}{}_{\lambda}$, and $\partial_{\mu}{}^{*}\partial^{\mu\nu}{}_{\lambda} = 0$.

$$\partial_\lambda A^\lambda = 0. \tag{31}$$

This can directly be shown as follows:

$$\partial_\lambda A^\lambda = \mu_0 \int \partial_\lambda G J^\lambda(x')\mathrm{d}^4x' = \mu_0 \int G\partial'_\lambda J^\lambda(x')\mathrm{d}^4x' = 0, \tag{32}$$

where the continuity equation $\partial'_\lambda J^\lambda = 0$; the identities $\partial_\lambda G = -\partial'_\lambda G$, $\partial'_\lambda(GJ^\lambda) = -\partial_\lambda GJ^\lambda + G\partial'_\lambda J^\lambda$; and the fact that the surface integral originated by $\partial'_\lambda(GJ^\lambda)$ vanishes at spatial infinity have been used. Similarly, if one takes the d'Alembert operator $\partial_\mu\partial^\mu \equiv -\Box^2$ to the four-potential in (30) then one obtains the wave equation

$$\partial_\mu\partial^\mu A^\lambda = \mu_0 \int \partial_\mu\partial^\mu G J^\lambda(x')\mathrm{d}^4x' = \mu_0 J^\lambda, \tag{33}$$

where the result $\partial_\mu\partial^\mu G = \delta^{(4)}(x - x')$ has been used.

Clearly, (26) and (30) describe the same retarded four-potential ($A_L^\nu = A^\nu$). Therefore the equivalence between the two covariant procedures to derive this retarded four-potential is established.

5. Conceptual and pedagogical advantages of the two procedures

The traditional procedure will be called the first procedure and the procedure based on the retarded Helmholtz theorem will be called the second procedure.

- The first procedure uses gauge invariance and then one is rapidly convinced that the scalar and vector potentials cannot have a physical meaning because they are ambiguous to describe the observed electric and magnetic fields. However, when the gauge is fixed through the adoption of the Lorenz gauge then one obtains well-defined wave equations for the potentials and if additionally one assumes causality and appropriate boundary conditions then one can solve these equations obtaining the retarded potentials. A demonstration of the uniqueness of the retarded potentials can be found in [16]. Put differently, the uniqueness of the retarded potentials is obtained by imposing the Lorenz-gauge condition, causality, and appropriate boundary conditions.
- In the first procedure one can fix another gauge condition (for example, the Kirchhoff gauge [17]: $\nabla \cdot \mathbf{A} - (1/c^2)\partial\Phi/\partial t = 0$) and then the propagation properties of the potentials will be different to those of the retarded potentials. However, as pointed out by Jackson [18]: '...whatever propagation or nonpropagation characteristics are exhibited by the potentials in a particular gauge, the electric and magnetic fields are always the same and display the experimentally verified properties of causality and propagation at the speed of light'.
- In the first procedure the starting point is the Maxwell's homogeneous equations: $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} + \partial\mathbf{B}/\partial t = 0$, which allow one to introduce the potentials Φ and $\mathbf{A}$ via $\mathbf{E} = -\nabla\Phi - \partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla \times \mathbf{A}$. However, a different situation occurs for Maxwell's equations with magnetic monopoles which are no longer homogeneous: $\nabla \cdot \mathbf{B} = \mu_0\rho_g$ and $\nabla \times \mathbf{E} + \partial\mathbf{B}/\partial t = -\mu_0\mathbf{J}_g$, where ρ_g and $\mathbf{J}_g$ are the magnetic charge and current densities. The first of these equations does not imply a potential $\mathbf{A}$ such that $\mathbf{B} = \nabla \times \mathbf{A}$. If one insists in considering the validity of this last equation in the presence of magnetic charges then the potential $\mathbf{A}$ must be singular in a defined region of space. This was the approach followed by Dirac to find its quantisation condition,

according to which the product of the electric and magnetic charges must be quantised in integer multiples of the smallest quantum of angular momentum, $\hbar/2$ [19]. Alternatively, Maxwell's equations with magnetic monopoles can be expressed in terms of non-singular potentials if one introduces by hand two pairs of potentials: $(\Phi_e, \mathbf{A}_e)$ and $(\Phi_g, \mathbf{A}_e)$.

- The second procedure based on the retarded Helmholtz theorem is relatively new in the literature, although the idea of introducing potentials by considering expressions for fields was recently discussed [10, 11]. In this second procedure one applies this theorem to the electric and magnetic fields of Maxwell's equations. The theorem guarantees causality, propagation at the speed of light and uniqueness of the electric and magnetic fields (provided they vanish sufficiently rapidly at infinity). In this procedure one 'extracts' by inspection the retarded potentials from the expressions for the retarded fields. *A posteriori* one verifies that these retarded potentials satisfy the same wave equations than those of the first procedure and also the Lorenz-gauge condition.
- The second procedure is appropriate to introduce the retarded potentials of electrodynamics with magnetic monopoles. The appendix shows how the retarded electric potentials $(\Phi_e, \mathbf{A}_e)$ and the retarded magnetic potentials $(\Phi_g, \mathbf{A}_g)$ can be easily introduced by inspection by considering the expressions for the retarded fields produced by electric and magnetic sources. The relation between fields and potentials is generalised by the following expressions

$$\begin{aligned} \mathbf{E} &= -\nabla\Phi_e - \nabla\times\mathbf{A}_g - \frac{\partial\mathbf{A}_e}{\partial t}, \\ \mathbf{B} &= -\nabla\Phi_g + \nabla\times\mathbf{A}_e - \frac{1}{c^2}\frac{\partial\mathbf{A}_g}{\partial t}. \end{aligned} \tag{34}$$

- The second procedure can be used as an alternative method to introduce the retarded potentials in an undergraduate course in electrodynamics. In fact, after addressing the first procedure, the instructor could optionally present the second procedure with the purpose of enlighten the concept of retarded potentials. As Richard Feynman pointed out [20] '... there is a pleasure in recognising old things from a new point of view. Also, there are problems for which the new point of view offers a distinct advantage'.
- As shown in section 4 both procedures can be expressed in the four-dimensional Minkowski space-time. In this covariant presentation both procedures use the retarded Green function of the wave equation. Therefore they are suitable for a graduate course in electrodynamics.

6. The instantaneous potentials of Maxwell's equations

According to the traditional procedure one can adopt the Coulomb-gauge condition

$$\nabla\cdot\mathbf{A}_C = 0, \tag{35}$$

in equations (5) and (6). Then they become

$$\nabla^2\Phi_C = -\frac{\rho}{\epsilon_0}, \quad \nabla^2\mathbf{A}_C = -\mu_0\mathbf{J} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}. \tag{36}$$

These equations are satisfied by the instantaneous potentials

$$\mathbf{A}_C(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int\frac{1}{R}\left(\mathbf{J}(\mathbf{r}', t) + \epsilon_0\frac{\partial \mathbf{E}(\mathbf{r}', t)}{\partial t}\right)d^3r', \tag{37}$$

$$\Phi_C(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}', t)}{R}d^3r'. \tag{38}$$

These potentials are rarely discussed in standard textbooks (Rosser in [3] pp 84–88 comments on these potentials). To show that a potential $\mathbf{A}$ satisfying the Coulomb-gauge can always be found, suppose that the original potential $\mathbf{A}$ satisfies the second equation in (36) but does not satisfy (35), i.e. $\nabla \cdot \mathbf{A} = g \neq 0$, where g is a known scalar function of space and time. Gauge invariance allows one to write the Poisson equation: $\nabla^2\Lambda = -g$, which guarantees the validity of the Coulomb-gauge condition for the transformed potential: $\nabla\cdot\mathbf{A}' = 0$. In fact, using the second transformation of (9) in the expression $\nabla\cdot\mathbf{A}'$ it follows that

$$\nabla \cdot\mathbf{A}' = \nabla \cdot (\mathbf{A} + \nabla\Lambda) = g + \nabla^2\Lambda = 0. \tag{39}$$

A second procedure allows one to find the instantaneous potentials by applying the standard Helmholtz theorem to Maxwell's equations. This theorem can be formulated for time-dependent vector fields. It states that any vector field $\mathbf{F}(\mathbf{r}, t)$ that goes to zero faster than $1/r$ as $r \to \infty$ can be expressed as [4, 21]:

$$\mathbf{F}(\mathbf{r}, t) = -\nabla\int\frac{\nabla' \cdot \mathbf{F}(\mathbf{r}, t)}{4\pi R}\,d^3r' + \nabla \times \int\frac{\nabla' \times \mathbf{F}(\mathbf{r}, t)}{4\pi R}\,d^3r'. \tag{40}$$

When (40) is applied to Maxwell's equations and an appropriate integration by parts is made, one obtains the instantaneous electric and magnetic fields

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\left\{\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}', t)}{R}d^3r'\right\} - \frac{\partial}{\partial t}\left\{\frac{\mu_0}{4\pi}\int\frac{1}{R}\left(\mathbf{J}(\mathbf{r}', t) + \epsilon_0\frac{\partial \mathbf{E}(\mathbf{r}', t)}{\partial t}\right)d^3r'\right\}, \tag{41}$$

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \left\{\frac{\mu_0}{4\pi}\int\frac{1}{R}\left(\mathbf{J}(\mathbf{r}', t) + \epsilon_0\frac{\partial \mathbf{E}(\mathbf{r}', t)}{\partial t}\right)d^3r'\right\}. \tag{42}$$

We then define the instantaneous potentials $\Phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ as

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int\frac{1}{R}\left(\mathbf{J}(\mathbf{r}', t) + \epsilon_0\frac{\partial \mathbf{E}(\mathbf{r}', t)}{\partial t}\right)d^3r', \tag{43}$$

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}', t)}{R}d^3r'. \tag{44}$$

The instantaneous potential $\mathbf{A}$ is seen to satisfy the equation

$$\nabla \cdot \mathbf{A} = 0. \tag{45}$$

In fact, from (43), Gauss' law, and the continuity equation it follows that

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \frac{\mu_0}{4\pi}\int \nabla \cdot \left(\frac{\mathbf{J}}{R}\right)d^3r' + \frac{1}{4\pi c^2}\frac{\partial}{\partial t}\int \nabla \cdot \left(\frac{\mathbf{E}}{R}\right)d^3r' \\ &= \frac{\mu_0}{4\pi}\int \frac{1}{R}\left(\nabla' \cdot \mathbf{J} + \frac{\partial\rho}{\partial t}\right)d^3r' = 0,\end{aligned} \tag{46}$$

where the identity $(\nabla'\cdot \mathbf{J})/R = \nabla \cdot (\mathbf{J}/R) + \nabla'\cdot(\mathbf{J}/R)$, and the fact that the surface integral originated by the term $\nabla'\cdot(\mathbf{J}/R)$ vanishes at space infinity have been used. Similarly, if one takes the Laplacian operator ∇^2 to the potentials in (43) and (44) then one obtains the Poisson equations

$$\nabla^2\Phi = \frac{1}{4\pi\epsilon_0}\int\nabla^2\left(\frac{\rho}{R}\right)\mathrm{d}^3r' = -\frac{\rho}{\epsilon_0}, \tag{47}$$

$$\nabla^2\mathbf{A} = \frac{\mu_0}{4\pi}\int\nabla^2\left(\frac{\mathbf{J}}{R}\right)\mathrm{d}^3r' + \frac{1}{4\pi c^2}\frac{\partial}{\partial t}\int\nabla^2\left(\frac{\mathbf{E}}{R}\right)\mathrm{d}^3r' = -\mu_0\mathbf{J} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}, \tag{48}$$

where the identity $\nabla^2(1/R) = -4\pi\delta(\mathbf{r} - \mathbf{r}')$ has been used.

As may be seen, the set of equations formed by (37) and (38) and the set formed by (43) and (44) describe the same instantaneous potentials ($\Phi_C = \Phi$ and $\mathbf{A}_C = \mathbf{A}$). Therefore the equivalence between the two procedures to derive these instantaneous potentials is established.

7. Instantaneous or retarded: a matter of taste?

The question in the title can be put in a historical perspective. Consider again the retarded potentials (7) in their explicit forms:

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}', t - R/c)}{R}\mathrm{d}^3r', \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}', t - R/c)}{R}\mathrm{d}^3r'. \tag{49}$$

The temporal connection between the potentials Φ and $\mathbf{A}$ and their sources the ρ and $\mathbf{J}$ is manifestly retarded, i.e. to determine the potentials at the observation time t one needs to know the sources at the retarded time $t - R/c$.

Consider now the instantaneous potentials given in (43) and (44):

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}', t)}{R}\mathrm{d}^3r', \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int\frac{1}{R}\left(\mathbf{J}(\mathbf{r}', t) + \epsilon_0\frac{\partial\mathbf{E}(\mathbf{r}', t)}{\partial t}\right)\mathrm{d}^3r'. \tag{50}$$

Maxwell called the 'true current' the sum of the conduction and displacement currents [22]: $\mathbf{J} + \epsilon_0\partial\mathbf{E}/\partial t$ and considered the latter as being 'electromagnetically equivalent' to the former. Notice the manifestly instantaneous connection between the potentials Φ and $\mathbf{A}$ and their sources ρ, $\mathbf{J}$ and $\epsilon_0\partial\mathbf{E}/\partial t$, i.e. to determine the potentials at the time t one needs to know their sources at the same time.

In 1894 Poincaré pointed out [23]: 'In calculating $\mathbf{A}$ (the potential given in (50)) Maxwell takes into account the currents of conduction [$\mathbf{J}$] and those of displacement [$\epsilon_0\partial\mathbf{E}/\partial t$]; and he supposes that the attraction takes place according to Newton's law, i.e. instantaneously. But in calculating [$\mathbf{A}$] in (49) on the contrary we take account only of conduction currents [$\mathbf{J}$] and we suppose the attraction is propagated with the velocity of light. It is a matter of indifference whether we make this hypothesis of a propagation in time and consider only the induction due to conduction currents, or whether like Maxwell, we retain the old law of instantaneous induction and consider both conduction and the displacement currents'. The lesson one should learn from Poincaré's statement is that there is a dual description of potentials satisfying Maxwell's equations. Both descriptions are formally correct but their interpretation and practical usefulness is different.

As already noted, the little-used equations in (50) describe an instantaneous connection between the potentials Φ and $\mathbf{A}$ and the sources ρ, $\mathbf{J}$ and $\epsilon_0\partial\mathbf{E}/\partial t$. While the densities ρ and $\mathbf{J}$ are specified independently from the potentials Φ and $\mathbf{A}$, the displacement current $\epsilon_0\partial\mathbf{E}/\partial t$ depends on the field $\mathbf{E} = -\nabla\Phi - \partial\mathbf{A}/\partial t$, i.e. on the potentials Φ and $\mathbf{A}$ themselves. According to the second equation in (50) to determine the field $\mathbf{A}$, one needs to know, besides the current density $\mathbf{J}$, the field $\mathbf{E}$ (or its time derivative) and according to $\mathbf{E} = -\nabla\Phi - \partial\mathbf{A}/\partial t$, one needs to know, besides Φ, the potential $\mathbf{A}$ (or its time derivative). Several authors have pointed out the little practical usefulness of the instantaneous potentials in (50). Jackson has emphasised [24]: 'With the time derivative of the electric field as part of the 'source', equation (second in (50)) is only a curiosity, not a useful tool for finding the fields from their sources'. Rosser [25] has claimed that the second equation in (50) 'is really a little bit of an illusion' and has stressed the circular argument involved in the determination of this equation. On the contrary, equations (49) describe a retarded connection between the potentials Φ and $\mathbf{A}$ and their true sources ρ and $\mathbf{J}$, which are specified independently from Φ and $\mathbf{A}$. But perhaps, from a physical point of view the more criticisable aspect of the instantaneous vector potential $\mathbf{A}$ in equation (50) is that it is determined by the displacement current which is a non-local source [26].

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Appendix. Magnetic monopoles and the retarded Helmholtz theorem

Consider Maxwell equations with magnetic monopoles:

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho_e, \tag{A.1}$$

$$\nabla\cdot\mathbf{B} = \mu_0\rho_g, \tag{A.2}$$

$$\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = -\mu_0\mathbf{J}_g, \tag{A.3}$$

$$\nabla\times\mathbf{B} - \epsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t} = \mu_0\mathbf{J}_e. \tag{A.4}$$

If (14) is applied to (A.1)–(A.4) and an appropriate integration by parts made, then one gets the retarded fields [5]:

$$\mathbf{E} = -\nabla\left\{\frac{1}{4\pi\epsilon_0}\int\frac{[\rho_e]}{R}\mathrm{d}^3r'\right\} - \nabla\times\left\{\frac{\mu_0}{4\pi}\int\frac{[\mathbf{J}_g]}{R}\mathrm{d}^3r'\right\} - \frac{\partial}{\partial t}\left\{\frac{\mu_0}{4\pi}\int\frac{[\mathbf{J}_e]}{R}\mathrm{d}^3r'\right\}, \tag{A.5}$$

$$\mathbf{B} = -\nabla\left\{\frac{\mu_0}{4\pi}\int\frac{[\rho_g]}{R}\mathrm{d}^3r'\right\} + \nabla\times\left\{\frac{\mu_0}{4\pi}\int\frac{[\mathbf{J}_e]}{R}\mathrm{d}^3r'\right\} - \frac{1}{c^2}\frac{\partial}{\partial t}\left\{\frac{\mu_0}{4\pi}\int\frac{[\mathbf{J}_g]}{R}\mathrm{d}^3r'\right\}. \tag{A.6}$$

From these expressions one can extract by inspection the retarded electric potentials (Φ_e, $\mathbf{A}_e$) and the retarded magnetic potentials (Φ_g, $\mathbf{A}_g$):

$$\Phi_e = \frac{1}{4\pi\epsilon_0}\int\frac{[\rho_e]}{R}\mathrm{d}^3r', \quad \mathbf{A}_e = \frac{\mu_0}{4\pi}\int\frac{[\mathbf{J}_e]}{R}\mathrm{d}^3r, \tag{A.7}$$

$$\Phi_g = \frac{\mu_0}{4\pi}\int\frac{[\rho_g]}{R}\mathrm{d}^3r', \quad \mathbf{A}_g = \frac{\mu_0}{4\pi}\int\frac{[\mathbf{J}_g]}{R}\mathrm{d}^3r'. \tag{A.8}$$

In terms of these potentials the equations (A.5) and (A.6) take the form

$$\mathbf{E} = -\nabla\Phi_e - \nabla \times \mathbf{A}_g - \frac{\partial \mathbf{A}_e}{\partial t}, \tag{A.9}$$

$$\mathbf{B} = -\nabla\Phi_g + \nabla \times \mathbf{A}_e - \frac{1}{c^2}\frac{\partial \mathbf{A}_g}{\partial t}. \tag{A.10}$$

The retarded potentials in (A.7) and (A.8) are seen to satisfy the equations

$$\nabla \cdot \mathbf{A}_e + \frac{1}{c^2}\frac{\partial \Phi_e}{\partial t} = 0, \quad \nabla \cdot \mathbf{A}_g + \frac{\partial \Phi_g}{\partial t} = 0. \tag{A.11}$$

In fact, using the potentials in (A.7) and (A.8) and the continuity equations for the electric and magnetic charges one obtains

$$\begin{aligned}\nabla \cdot \mathbf{A}_e + \frac{1}{c^2}\frac{\partial \Phi_e}{\partial t} &= \frac{\mu_0}{4\pi}\int \nabla \cdot \left(\frac{[\mathbf{J}_e]}{R}\right) \mathrm{d}^3r' + \frac{1}{4\pi\epsilon_0 c^2}\int \frac{\partial}{\partial t}\left(\frac{[\rho_e]}{R}\right)\mathrm{d}^3r' \\ &= \frac{\mu_0}{4\pi}\int \frac{1}{R}\left[\nabla' \cdot \mathbf{J}_e + \frac{\partial \rho_e}{\partial t'}\right]\mathrm{d}^3r' = 0,\end{aligned} \tag{A.12}$$

$$\begin{aligned}\nabla \cdot \mathbf{A}_g + \frac{\partial \Phi_g}{\partial t} &= \frac{\mu_0}{4\pi}\int \nabla \cdot \left(\frac{[\mathbf{J}_g]}{R}\right) \mathrm{d}^3r' + \frac{\mu_0}{4\pi}\int \frac{\partial}{\partial t}\left(\frac{[\rho_g]}{R}\right)\mathrm{d}^3r' \\ &= \frac{\mu_0}{4\pi}\int \frac{1}{R}\left[\nabla' \cdot \mathbf{J}_g + \frac{\partial \rho_g}{\partial t'}\right]\mathrm{d}^3r' = 0,\end{aligned} \tag{A.13}$$

where again the identities $[\nabla' \cdot \mathbf{J}_e]/R = \nabla \cdot ([\mathbf{J}_e]/R) + \nabla' \cdot ([\mathbf{J}_e]/R)$, $[\nabla' \cdot \mathbf{J}_g]/R = \nabla \cdot ([\mathbf{J}_g]/R) + \nabla' \cdot ([\mathbf{J}_g]/R)$, $[\partial\rho_e/\partial t'] = \partial[\rho_e]/\partial t$ and $[\partial\rho_g/\partial t'] = \partial[\rho_g]/\partial t$ have been used, and also the surface integrals originated by the terms $\nabla' \cdot ([\mathbf{J}_e]/R)$ and $\nabla' \cdot ([\mathbf{J}_g]/R)$ vanish at space infinity. Similarly, if one takes the d'Alembert operator $\Box^2$ to the potentials in (A.7) and (A.8) then one gets the set of four wave equations

$$\Box^2\Phi_e = \frac{1}{4\pi\epsilon_0}\int \Box^2\left(\frac{[\rho_e]}{R}\right)\mathrm{d}^3r' = -\frac{\rho_e}{\epsilon_0}, \tag{A.14}$$

$$\Box^2\mathbf{A}_e = \frac{\mu_0}{4\pi}\int \Box^2\left(\frac{[\mathbf{J}_e]}{R}\right)\mathrm{d}^3r' = -\mu_0\mathbf{J}_e, \tag{A.15}$$

$$\Box^2\Phi_g = \frac{\mu_0}{4\pi}\int \Box^2\left(\frac{[\rho_g]}{R}\right)\mathrm{d}^3r' = -\mu_0\rho_g, \tag{A.16}$$

$$\Box^2\mathbf{A}_g = \frac{\mu_0}{4\pi}\int \Box^2\left(\frac{[\mathbf{J}_g]}{R}\right)\mathrm{d}^3r' = -\mu_0\mathbf{J}_g, \tag{A.17}$$

where the identity $\Box^2([...]/R) = -4\pi[...]\delta(\mathbf{r} - \mathbf{r}')$ has been used.

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