Lorentz transformations and the wave equation

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Abstract. In this note we explicitly show how the Lorentz transformations can be derived by demanding form invariance of the d’Alembert operator in inertial reference frames.

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1. Introduction

The homogeneous scalar wave equation is form invariant under the Lorentz transformations [1]. It is therefore reasonable to expect that these transformations may be derived by requiring form invariance of the homogeneous scalar wave equation. Although this expectation turns out to be correct, its explicit demonstration does not seem to be widely known [2]. In this note we derive the Lorentz transformations in their standard configuration by demanding form invariance of the d’Alembert operator in inertial reference frames [3]. This relatively simple derivation of the Lorentz transformations could be suitable for undergraduate courses of electromagnetism and special relativity.

2. Lorentz transformations from the invariance of the d’Alembertian

Consider the standard configuration in which two inertial frames $S$ and $S'$ are in relative motion with speed $v$ along their common $xx'$ direction. The origins of the two frames coincide at the instant $t=t'=0$. The coordinates transverse to the relative motion of the frames $S$ and $S'$ are assumed to be invariant: $y'=y$ and $z'=z$. The corresponding derivative operators are also assumed to be invariant: $\partial/\partial y = \partial/\partial y'$ and $\partial/\partial z = \partial/\partial z'$ which imply $\partial^2/\partial y^2 = \partial^2/\partial y'^2$ and $\partial^2/\partial z^2 = \partial^2/\partial z'^2$. Our problem reduces then to find the transformation rules for $\partial^2/\partial x^2$ and $\partial^2/\partial t^2$. The invariance of the reduced d’Alembert operator can be expressed as

$$\frac{\partial^2}{\partial x'^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t'^2} = \frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}. \quad (1)$$

We observe that in stating this invariance, we are assuming the postulate of relativity, which states that the laws of physics are the same in all inertial frames and the postulate of the constancy of the speed of light, which states that the speed of light in vacuum has the same value in all inertial frames. Equation (1) can be factored as follows

$$\left( \frac{\partial}{\partial x} - \frac{1}{c} \frac{\partial}{\partial t} \right) \left( \frac{\partial}{\partial x'} + \frac{1}{c} \frac{\partial}{\partial t'} \right) = \left( \frac{\partial}{\partial x'} - \frac{1}{c} \frac{\partial}{\partial t'} \right) \left( \frac{\partial}{\partial x} + \frac{1}{c} \frac{\partial}{\partial t} \right). \quad (2)$$
By assuming linearity for the involved transformations of operators, we can write

\[
\frac{\partial}{\partial x} - \frac{1}{c} \frac{\partial}{\partial t} = A \left( \frac{\partial}{\partial x'} - \frac{1}{c} \frac{\partial}{\partial t'} \right) \tag{3}
\]

\[
\frac{\partial}{\partial x} + \frac{1}{c} \frac{\partial}{\partial t} = A^{-1} \left( \frac{\partial}{\partial x'} + \frac{1}{c} \frac{\partial}{\partial t'} \right) \tag{4}
\]

where the factor \( A \) is independent of the derivative operators but can depend on the velocity \( v \).

In order to determine \( A \), we demand that the expected linear transformation relating primed and unprimed time-derivative operators should appropriately reduce to the corresponding Galilean transformation \([4]\): \( \partial/\partial t = \partial/\partial t' - \gamma v/\partial x' \). Our demand is consistent with a linear transformation of the general form: \( \partial/\partial t = F(v) \left( \partial/\partial t' - \gamma v/\partial x' \right) \), where \( F(v) \) depends on the velocity \( v \) so that \( F(v) \rightarrow 1 \) when \( v << c \). From this general transformation it follows that if \( \partial/\partial t = 0 \) then \( \partial/\partial t' = \gamma v/\partial x' \) because \( F(v) \neq 0 \). Using this result in Eqs. (3) and (4) we obtain

\[
\frac{\partial}{\partial x} = A \left( \frac{\partial}{\partial x'} - \frac{\gamma v}{c} \frac{\partial}{\partial t'} \right) \tag{5}
\]

\[
\frac{\partial}{\partial x} = A^{-1} \left( \frac{\partial}{\partial x'} + \frac{\gamma v}{c} \frac{\partial}{\partial t'} \right) \tag{6}
\]

By combining these equations we get the expressions

\[
A = \left( \frac{1 + v/c}{1 - v/c} \right)^{1/2} = \gamma \left( 1 + v/c \right) \tag{7}
\]

\[
A^{-1} = \left( \frac{1 - v/c}{1 + v/c} \right)^{1/2} = \gamma \left( 1 - v/c \right) \tag{8}
\]

where \( \gamma = 1/\sqrt{1-v^2/c^2} \). Using Eqs. (7) and (8) into Eqs. (3) and (4), we obtain the transformation laws \([5]\):

\[
\frac{\partial}{\partial x} - \frac{1}{c} \frac{\partial}{\partial t} = \gamma \left( 1 + \frac{v}{c} \right) \left( \frac{\partial}{\partial x'} - \frac{1}{c} \frac{\partial}{\partial t'} \right) \tag{9}
\]

\[
\frac{\partial}{\partial x} + \frac{1}{c} \frac{\partial}{\partial t} = \gamma \left( 1 - \frac{v}{c} \right) \left( \frac{\partial}{\partial x'} + \frac{1}{c} \frac{\partial}{\partial t'} \right) \tag{10}
\]

By adding and subtracting these equations, we obtain the transformation laws connecting unprimed and primed derivative operators

\[
\frac{\partial}{\partial x} = \gamma \left( \frac{\partial}{\partial x'} - \frac{v}{c^2} \frac{\partial}{\partial t'} \right) \tag{11}
\]

\[
\frac{\partial}{\partial t} = -\gamma v \left( \frac{\partial}{\partial x'} - \frac{v}{c^2} \frac{\partial}{\partial t'} \right) \tag{12}
\]

which must be completed with the transformations \( \partial/\partial y = \partial/\partial y' \) and \( \partial/\partial z = \partial/\partial z' \). These transformation laws are the Lorentz transformations for derivative operators of the standard configuration. We note that Eqs. (11) and (12) imply coordinate transformations of the form:

\( x' = x'(x,t), y' = y, z' = z \) and \( t' = t' (x,t) \). To find the explicit form of these transformations we can use Eqs. (11) and (12) to obtain

\[
\frac{\partial x'}{\partial x} = \gamma, \frac{\partial x'}{\partial t} = -\gamma v, \frac{\partial t'}{\partial t} = \gamma, \frac{\partial t'}{\partial x} = -\gamma v/c^2 \tag{13}
\]
The first relation in Eq. (13) implies (I): \( x' = \gamma x + f_1(t) \), where \( f_1(t) \) can be determined (up to a constant) by differentiating (I) with respect to the time \( t \) and using the second relation in Eq. (13): \( \frac{\partial x'}{\partial t} = \frac{d f_1(t)}{dt} = -\gamma v \). This last equality implies (II): \( f_1(t) = -\gamma vt + x_0 \), where \( x_0 \) is a constant. From (I) and (II) we obtain
\[
 x' = \gamma (x - vt) + x_0. \tag{14}
\]
The third relation in Eq. (13) implies (III): \( t' = \gamma t + f_2(x) \), where \( f_2(x) \) can be obtained (up to a constant) from differentiating (III) with respect to \( x \) and using the last relation in Eq. (13): \( \frac{\partial t'}{\partial x} = \frac{d f_2(x)}{dx} = -\gamma v/c^2 \). This last equality implies (IV): \( f_2(x) = -\gamma vx/c^2 + t_0 \), where \( t_0 \) is a constant. From (III) and (IV) we obtain
\[
 t' = \gamma (t - vx/c^2) + t_0. \tag{15}
\]
The origins of the frames \( S \) and \( S' \) coincide at the time \( t = t' = 0 \) and therefore we have \( x_0 = 0 \) and \( t_0 = 0 \). In this way we obtain the Lorentz transformations of the standard configuration:
\[
 x' = \gamma (x - vt), \quad t' = \gamma (t - vx/c^2), \tag{16}
\]
which must be completed with the remaining transformations: \( y' = y \) and \( z' = z \).

3. Conclusion

There is no doubt that the wave equation is one of the most important equations in physics. This equation describes a great variety of physical phenomena. Therefore the symmetries associated with the wave equation are equally important. Perhaps the most famous of these symmetries is the Lorentz symmetry. The d’Alembert operator, the basic ingredient of the wave equation, is shown to be form invariant under the Lorentz transformations. In this note we have traveled the inverse route and demanded form invariance of the d’Alembert operator to obtain the Lorentz transformations in their standard configuration. This derivation relies on the postulates of special relativity; it enriches the list of the many derivations of the Lorentz transformations presented over the years [6].

References

[1] It is well-known that the wave equation is invariant under the Lorentz transformations but the explicit demonstration of this statement is not usually presented in standard textbooks. Some popular undergraduate textbooks like Wangsness’s Electromagnetic Fields [2nd ed. (Wiley, New York 1986) p. 496] and also some graduate textbooks like Jackson’s Classical Electrodynamics [3rd ed. (Wiley, New York, 1999) p. 516] or Zangwill’s Modern Electrodynamics [(Cambridge University Press, Cambridge, 2012) p. 824] explicitly show that the homogeneous wave equation is not form invariant under the Galilean transformations but they do not explicitly demonstrate the form invariance of this equation under the Lorentz transformations. In the book Relativity and Gravitation by Tournèrc P. (Cambridge University Press, Cambridge, pp. 24-26), we find the explicit demonstration of the invariance of the wave equation under the Lorentz transformations.

[2] At first sight, the derivation of the Lorentz transformations from the form invariance of the wave equation seems to be a natural task. However, the present author has not been able to find some textbook in which such a derivation is presented.

[3] The present derivation of the Lorentz transformations is inspired in that appearing in the paper: Parker L and Schmieg G M 1970 Special Relativity and Diagonal Transformations Am. J. Phys. 38, 218-222. See, also, the subsequent paper of these authors: Parker L and Schmieg G M 1970 A Useful Form of the Minkowski Diagram Am. J. Phys. 38, 1298-1302. In these papers, Parker and Schmieg emphasize the now known as the light-cone variables, which were introduced by Dirac in his paper: Dirac P A M 1949 Forms of Relativistic Dynamics Rev. Mod. Phys. 21, 392-399. A pedagogical discussion of the light-cone coordinate system is given in the paper: Kim Y S and Noz M E 1982 Dirac’s light-cone coordinate system Am. J. Phys. 50, 721-724.

[4] See, for example, Wagness R K in Ref. 1 p. 496.
If we define the derivative operators:

\[ \partial_\xi \equiv \partial/\partial x - (1/c)\partial/\partial t, \partial_\eta \equiv \partial/\partial x + (1/c)\partial/\partial t, \partial_\xi' \equiv \partial/\partial x' - (1/c)\partial/\partial t', \]

and

\[ \partial_\eta' \equiv \partial/\partial x' + (1/c)\partial/\partial t', \]

then the set formed by the transformations in Eqs. (9) and (10) together with the transformations \( \partial/\partial y = \partial/\partial y' \) and \( \partial/\partial z = \partial/\partial z' \) can compactly be expressed in the tensor form \( \partial A = D B A \partial B' \), where

\[ \partial A = [\partial_\xi, \partial_\eta, \partial_y, \partial_z] \]

and

\[ \partial B' = [\partial_\xi', \partial_\eta', \partial_y', \partial_z'] \]

(summation on the repeated index \( B \) is understood). The non-vanishing elements of the transformation matrix \( D_B A \) are given by

\[ D_1^1 = A, D_2^2 = A^{-1}, D_3^3 = 1, D_4^4 = 1 \]

with \( A = \gamma(1 + v/c) \) and \( A^{-1} = \gamma(1 - v/c) \). This means that the matrix \( D \) is diagonal and therefore we can say, following Parker and Schmieg in Ref. 3, that \( \partial A = D_B A \partial B' \) represents a diagonal form of the Lorentz transformations (standard configuration) for the specified derivative operators.